132843

SEMIANNUAL STATUS REPORT

GRANT NUMBER NGR 41-001-036

SUBMITTED TO THE

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

BY

H. J. HELGERT

DEPARTMENT OF ELECTRICAL AND COMPUTER ENGINEERING

CLEMSON UNIVERSITY

CLEMSON, SOUTH CAROLINA 29631

JULY 1973

Statement of Research Accomplished

During the period December 15, 1972 to June 15, 1973 all work was concentrated on the definition and evaluation of a measure of performance for code-decoder combinations. The following results were obtained:

1. The definition of a performance measure for both block and convolutional codes which takes into account the amount of redundancy in the code, the amount of data rejected by the decoder, the accuracy of the data after decoding and the relative importance of the last two factors.
2. The evaluation of the above performance measure for two types of Hamming codes of block length n = 7, 15, 31, 63, 127, 255, 511, 1023 used over the binary symmetric channel and decoded by various algorithms.
3. A graphical comparison of the performance of these codes and an uncoded transmission system, as a function of the channel signal-to-noise ratio.

A detailed description of these results is appended.

On the Performance of Block Codes*

by

H. J. Helgert

Department of Electrical
and Computer Engineering

Clemson University

Clemson, South Carolina 29631

*This work was supported by the National Aeronautics and Space Administration under Grant Number NGR 41-001-036.

I. Introduction and Summary

Although the use of error control coding techniques in digital space communication systems has become fairly routine in recent years, there still exists a great deal of uncertainty as to the actual effectiveness of coding in achieving more reliable communication. The reason for this is to be found in the fact that the commonly used performance parameters do not take into account all the pertinent aspects of the coded transmission system. Thus, for example, the widely used Probability of Word Error criterion totally ignores the possibility that the decoder may incorporate some degree of data rejection. Likewise, the minimum distance criterion, another popular measure of code performance, is completely independent of the decoding algorithm and several other important system factors.

As a consequence of this state of affairs, it is virtually impossible to compare, say, a coding system with error correction and data rejection to one with error correction alone, using any of the existing criteria of performance, and it is therefore of value to define and evaluate measures which incorporate most, if not all, of the quantities affecting the overall system reliability. This is the objective of the present work.

II. Definition of Performance Measure

For the simple types of block codes normally employed in space communication systems, the complexity of the encoder and decoder is of little consequence, since the use of integrated circuit technology allows the construction of the basic components in an inexpensive fashion. Furthermore, the complexity is essentially independent of the particular code-decoder used.

The processing speed is generally a function of the type of logic used and the technology in the construction of the integrated circuits.

Although one could probably obtain cost figures as a function of processing speed, the importance of these costs in the overall system considerations is difficult to assess. Also, as with complexity, processing speed is not a strong function of the code-decoder combination.

Thus, the important factors determining the overall coding system performance are:

1. The accuracy of the data after decoding,
2. The amount of data rejected by the decoder,
3. The amount of redundancy in the code, and
4. The relative importance of data accuracy, data rejection, and data transmission rate.

Let us consider a situation in which N blocks of received digits from a binary (n, k) block code are to be decoded. The decoder generally rejects N-X blocks, leaving X blocks after decoding, of which Y are correct. (see Figure 1).

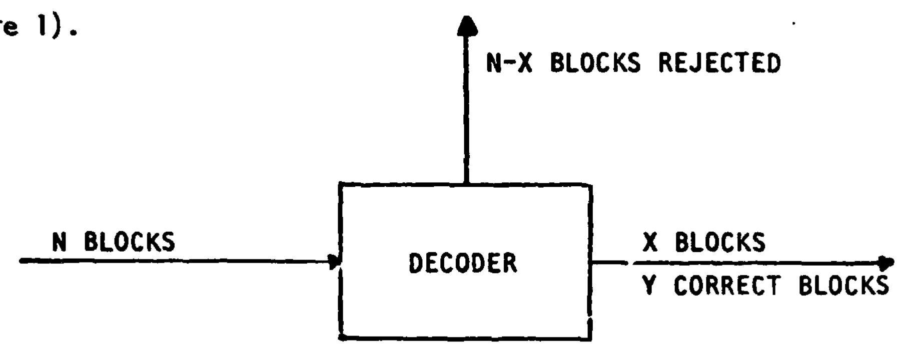


FIGURE 1. GENERAL DECODER CONFIGURATION

The amount of redundancy in the code is measured by the quantity

$$F_1 = \frac{k}{n} = \frac{\text{number of data digits per block}}{\text{total number of digits per block}} ,$$

the amount of data rejected by the decoder is measured by the quantity

$$F_2 = \frac{1}{N} E\{X\} ,$$

and the accuracy of the data after decoding is measured by the quantity

$$F_3 = \frac{E\{Y\}}{E\{X\}} .$$

Here E is the usual expectation operator.

We also define a quantity $0 \leq \alpha \leq 1$ which measures the relative importance of data accuracy and data rejection.

As an overall measure of performance of the code-decoder combination, we then take the function

$$F = F_1 \{\alpha F_2 + (1-\alpha) F_3\} .$$

When the N blocks are transmitted independently of each other and are treated as such by the decoder, F_2 reduces to the prob.. ility of word rejection. For a decoder with no data rejection, F_3 becomes the probability of correct decoding. The quantity α depends on the particular circumstances and is normally determined by the nature of the data. Small values of α correspond to a heavy emphasis on post decoding data accuracy, whereas large values of α imply emphasis on low rejection rates. In the next section, we evaluate F for a number of specific codes used over the binary symmetric channel.

III. Evaluation of F for Hamming Type Block Codes

We assume that N blocks are transmitted independently and with equal probability over a binary symmetric channel whose digit error probability is $p = 1-q$. The codes of interest are of two types: the standard (n, k) Hamming code described by the parity check matrix

$$H = \begin{bmatrix} 0 & 0 & . & . & . & 1 \\ 0 & 0 & . & . & . & 1 \\ . & & & & & . \\ . & & & & & . \\ 0 & 1 & . & . & . & 1 \\ 0 & 0 & . & . & . & 1 \end{bmatrix}$$

whose columns are all 2^m-1 nonzero binary m-tuples (m any integer greater than 2), and a modified Hamming code whose parity check matrix differs from the above only in having an additional row of ones on top. The first

code has block length $n = 2^m - 1$, $k = n-m$ information digits and minimum distance 3 and is thus able to correct all single errors. The second code has the same block length, $k = n-m-1$ and minimum distance 4 and can be decoded in either a single-error-correcting, double-error-detecting mode or a triple-error-detecting mode.

For both codes, the first step in decoding a received block $v = (v_1, v_2, \ldots v_n)$ consists of determining its syndrome. This is a binary (n-k)-typle given by

$$s = vH^T$$

where T denotes matrix transposition and the multiplication and addition operations are modulo 2.

We now consider four cases, including, for purposes of comparison, the uncoded transmission of data blocks of length n.

Case 1. No Coding - (n, n) Code

Decoding Rule: Pass every block unchanged

Evidently, $X = E\{X\} = N$ and since a block is correct at the decoder output if and only if it is correct at the input, we have $E\{Y\} = Nq^n$

Hence $F_1 = 1$; $F_2 = 1$; $F_3 = q^n$

and

$$F = \alpha + (1-\alpha)q^n$$

Case 2. Single-Error-Correcting Hamming Code

Decoding Rule: If the syndrome is zero, pass the block. If the syndrome is not equal to zero, assume a single error has occurred, correct it, and then pass the block.

Again, $E\{X\} = N$. For $E\{Y\}$ we have

$$E\{Y\} = N\{\text{Probability that a block has no error or a single error before decoding}\}$$

$$= N\{q^n + nq^{n-1}\, p\}$$

Therefore, $F_1 = \frac{n-m}{n}$, $F_2 = 1$, $F_3 = q^n + nq^{n-1}\,p$

and

$$F = \frac{n-m}{n}\ \{\alpha + (1-\alpha)(q^n + nq^{n-1}\,p)\}$$

Case 3. Single-Error-Correcting, Double-Error-Detecting Hamming Code

Decoding Rule: If the syndrome is zero, pass the block. If the first digit and at least one of the remaining digits in the syndrome are one, assume a single error has occurred, correct it, and then pass the block. For all other syndromes, reject the block.

We have $F_1 = \frac{n-m-1}{n}$,

$$F_2 = \frac{E\{X\}}{N} = \{\text{Probability that a block has zero syndrome or the first and at least one of the remaining digits equal one}\}$$

$$= \sum_{i=0}^{\frac{n-1}{2}} \{A_{2i}\, q^{n-2i}\, p^{2i} + [\binom{n}{2i+1} - A_{2i+1}]\, q^{n-2i+1}\, p^{2i+1}\}$$

where A_j is the number of codewords of weight j of the Single-Error-Correcting Hamming Code,

and for F_3 we obtain

$$F_3 = \frac{E\{Y\}}{E\{X\}} = \frac{1}{F_2}\ \{\text{Probability that a received block is correct or has a single error}\}$$

Case 4. Triple-Error-Detecting Hamming Code

Decoding Rule: If the syndrome is zero, pass the block. Otherwise, reject the block.

Here $F_1 = \frac{n-m-1}{n}$,

$$F_2 = \frac{E\{X\}}{N} = \{\text{Probability that a block has zero syndrome}\}$$

$$= \sum_{i=0}^{\frac{n-1}{2}} A_{2i}\, q^{n-2i}\, p^{2i}\ ,$$

and

$$F_j = \frac{E\{Y\}}{E\{X\}} = \frac{q^n}{F_2}$$

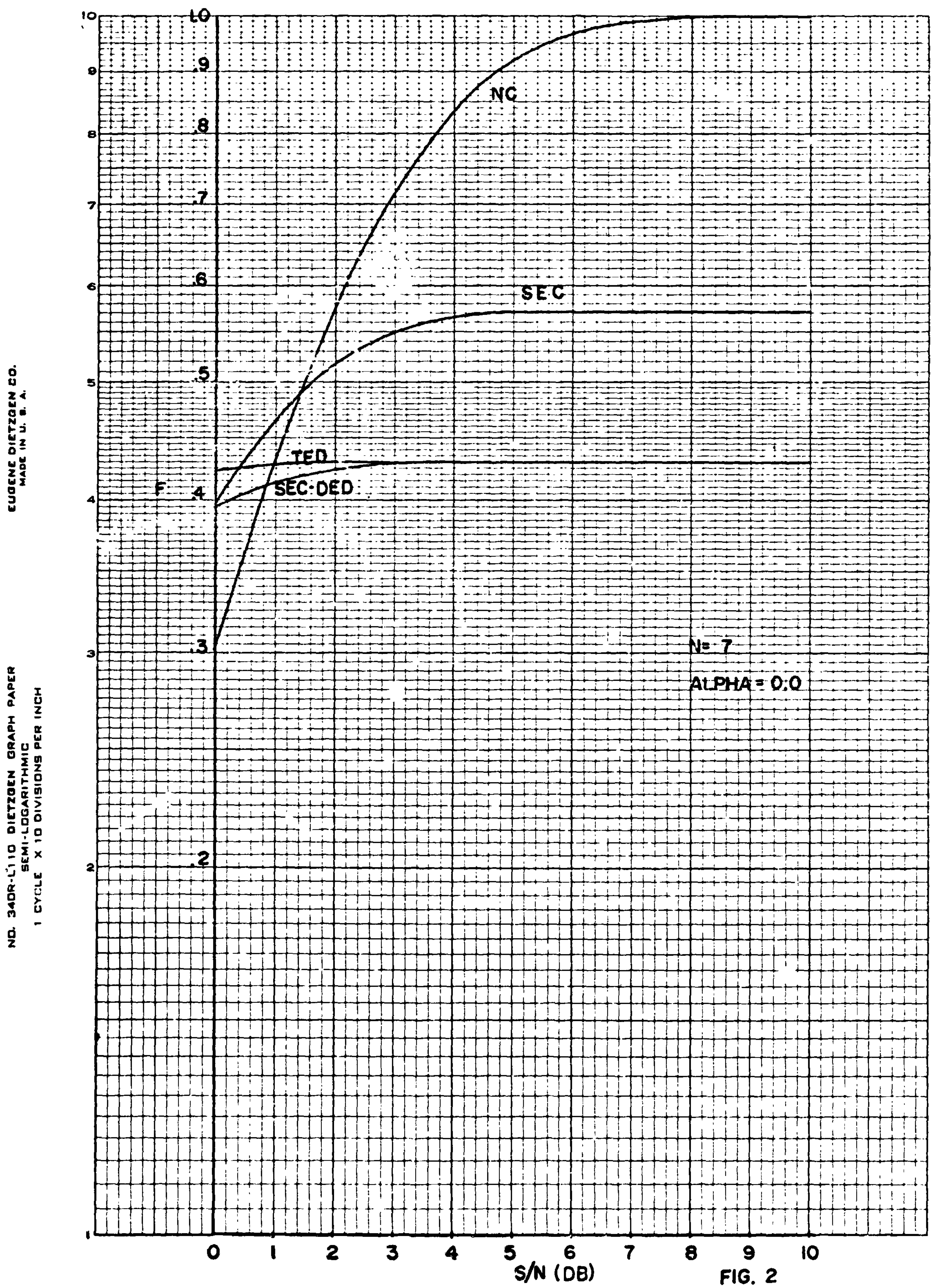
The Hamming code weight spectra required for Cases 3 and 4 may be obtained as the coefficients of the polynomial.

$$f(x) = \frac{1}{n+1} \{(1+x)^n + n(1+x)^{\frac{n-1}{2}} (1-x)^{\frac{n+1}{2}}\}$$

where A_i is the coefficient of x^i.

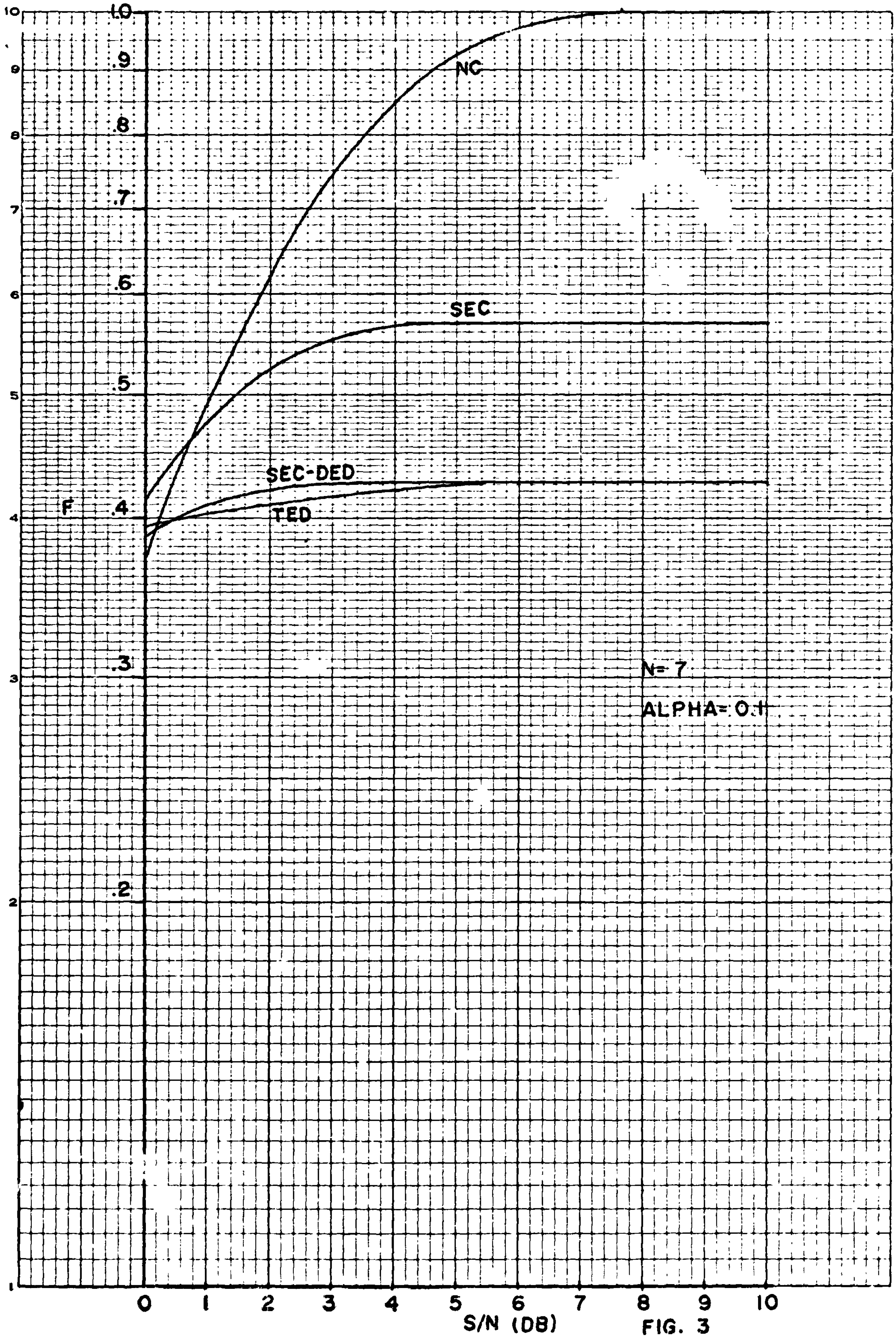
IV. Numerical Results

In Figures 2-49, we have plotted the performance measure F as a function of the signal-to-noise ratio of the binary symmetric channel in db, for all four cases described above, and for n = 7, 15, 31, 63, 127, 255, 511, 1023, and α = 0, 0.1, 0.3, 0.5, 0.8, 1.0.



NO. 340R-L110 DIETZGEN GRAPH PAPER
SEMI-LOGARITHMIC
1 CYCLE X 10 DIVISIONS PER INCH
EUGENE DIETZGEN CO.
MADE IN U. S. A.
NC
SEC
TED
SEC-DED
F
N= 7
ALPHA = 0.0
S/N (DB)

FIG. 2



1.0
.9
.8
.7
.6
.5
.4
.3
.2
F
NC
SEC
SEC-DED
TED
N=7
ALPHA=0.1
0
1
2
3
4
5
6
7
8
9
10
S/N (DB)

FIG. 3

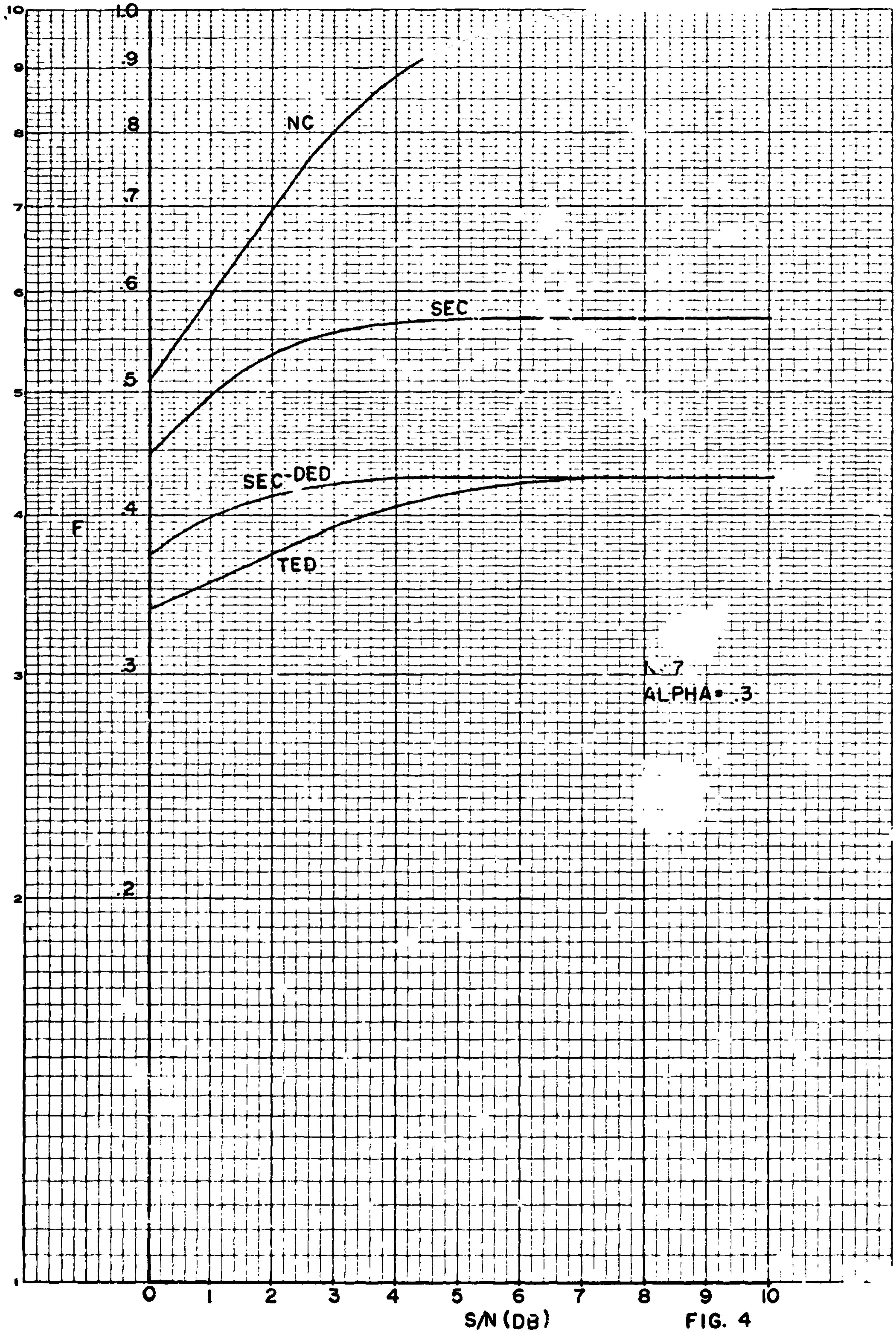


FIG. 4

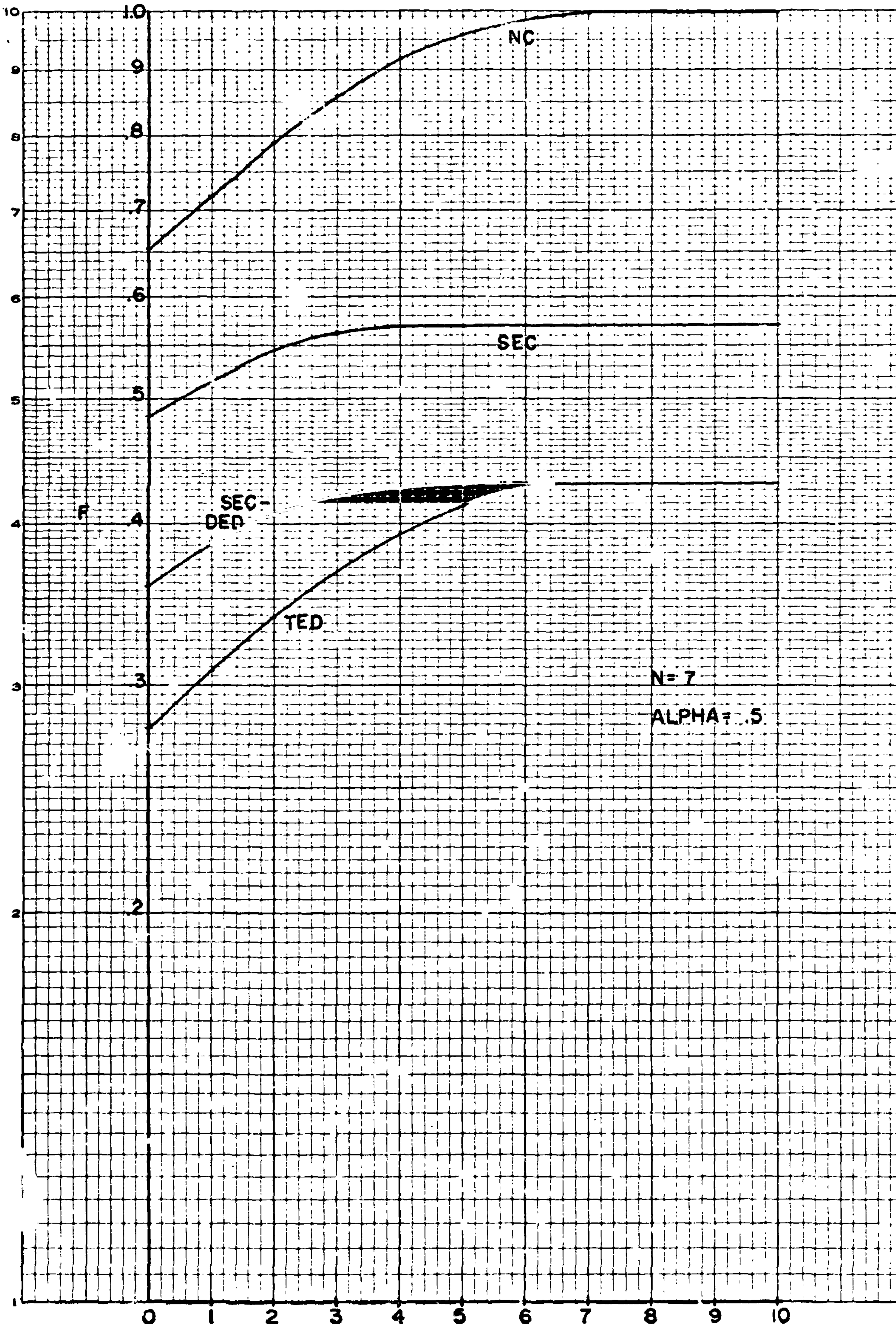


FIG. 5

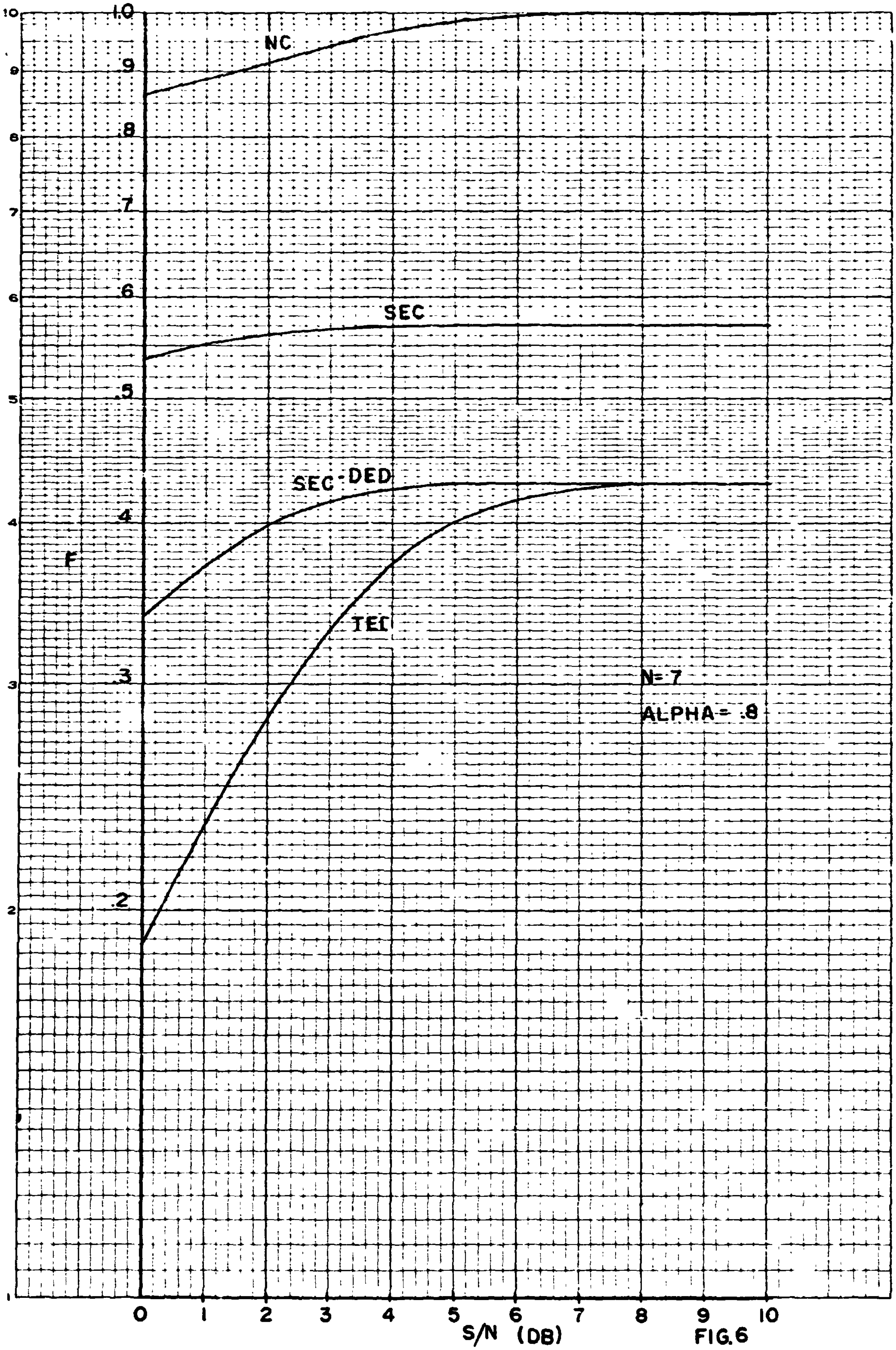


FIG.6

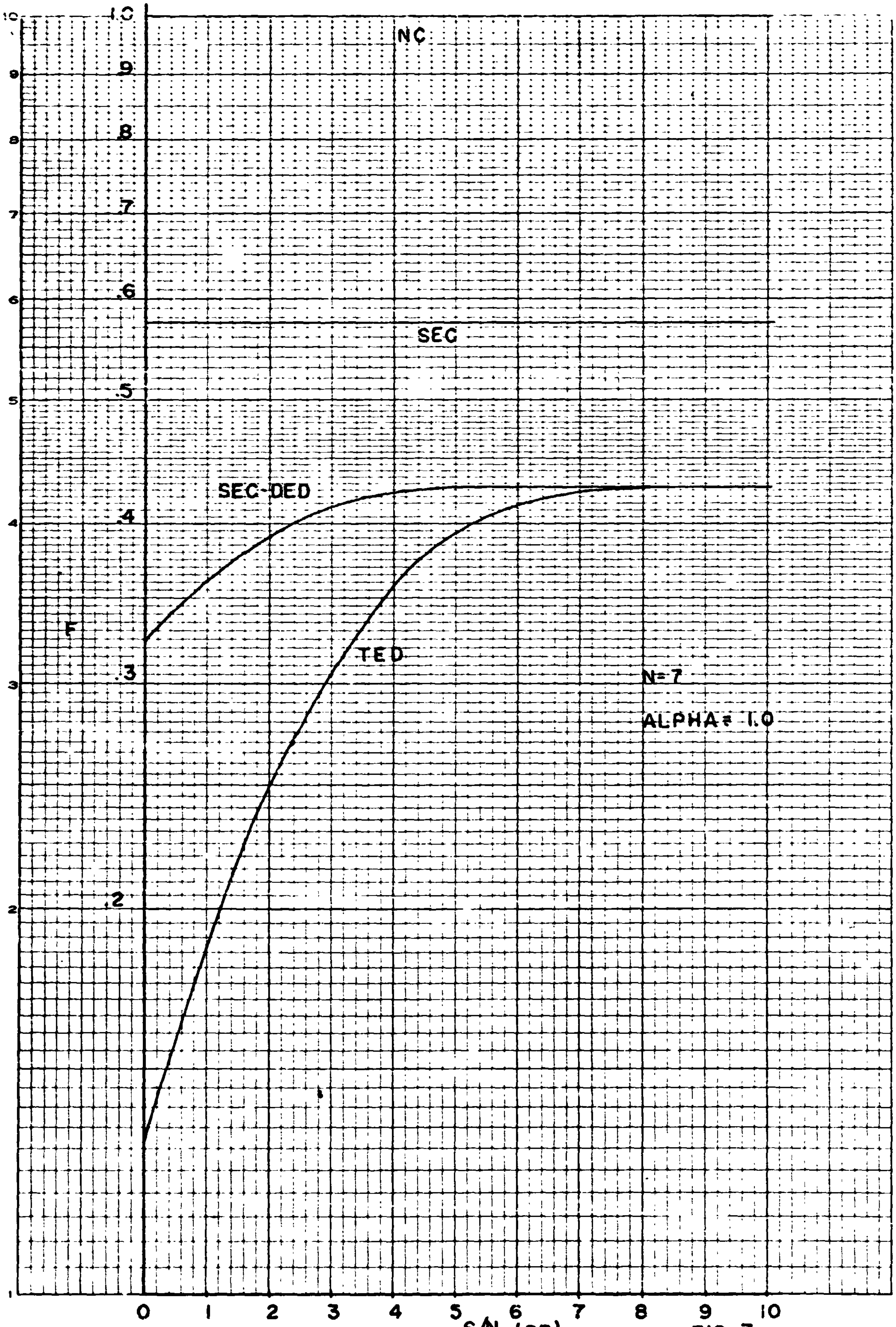
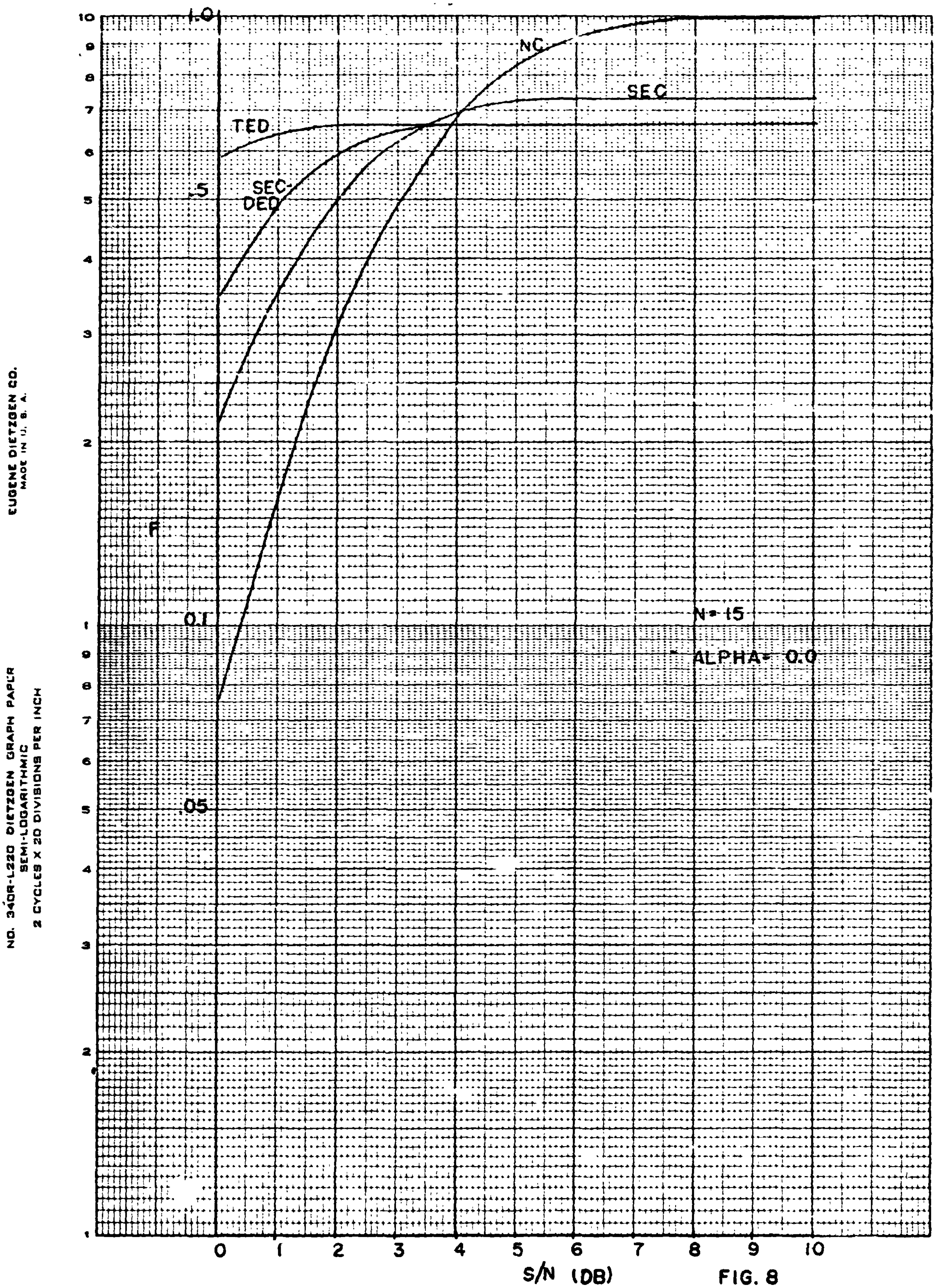


FIG. 7



1.0
NC
SEC
TED
.5
SEC-DED
F
0.1
N= 15
ALPHA= 0.0
.05
0
1
2
3
4
5
6
7
8
9
10
S/N (DB)
EUGENE DIETZGEN CO.
MADE IN U. S. A.
NO. 340R-L220 DIETZGEN GRAPH PAPER
SEMI-LOGARITHMIC
2 CYCLES X 20 DIVISIONS PER INCH

FIG. 8

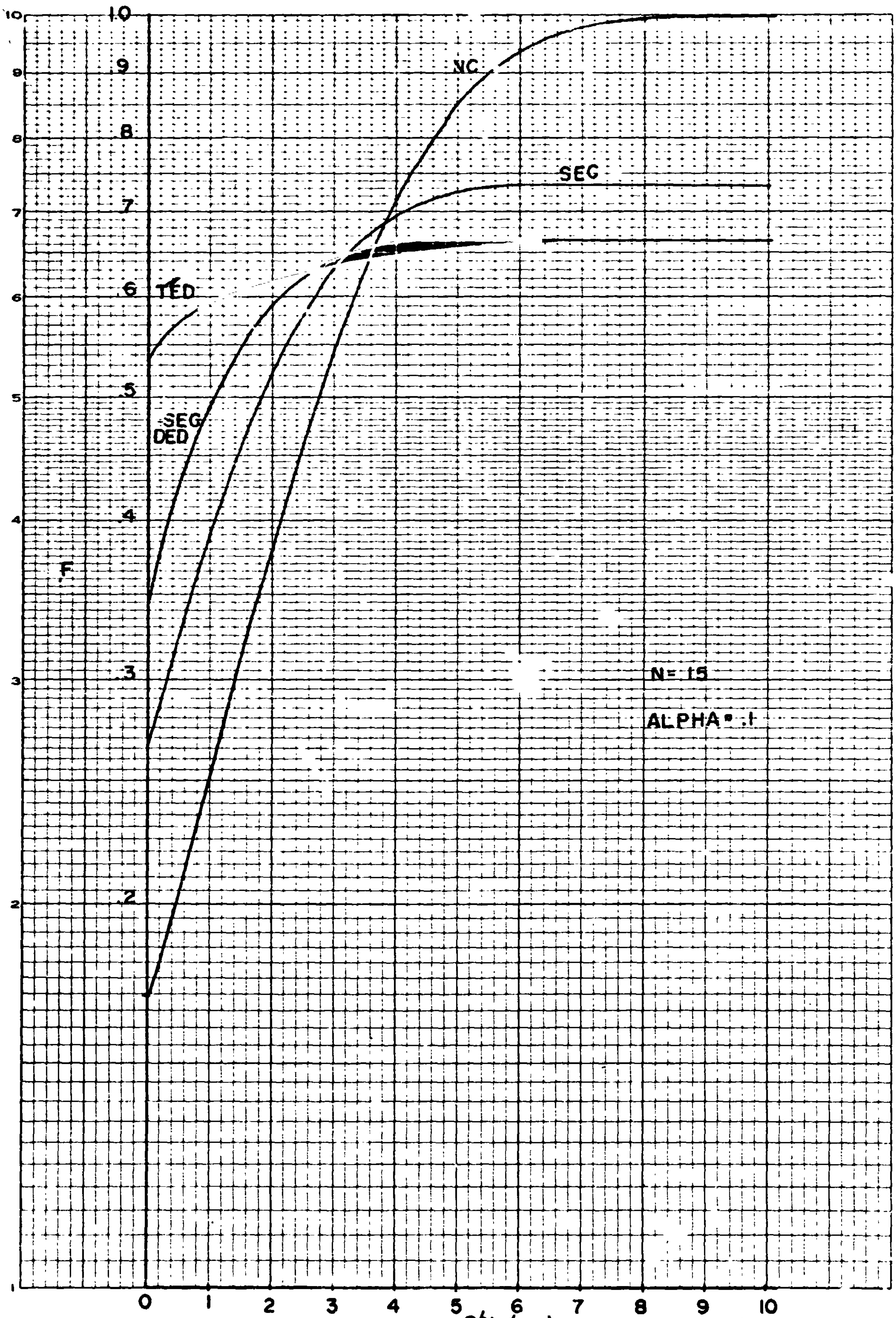

F
1.0
.9
.8
.7
.6
.5
.4
.3
.2
NC
SEG
TED
SEG
DED
N= 15
ALPHA= .1
0
1
2
3
4
5
6
7
8
9
10
S/N (DB)

FIG. 9

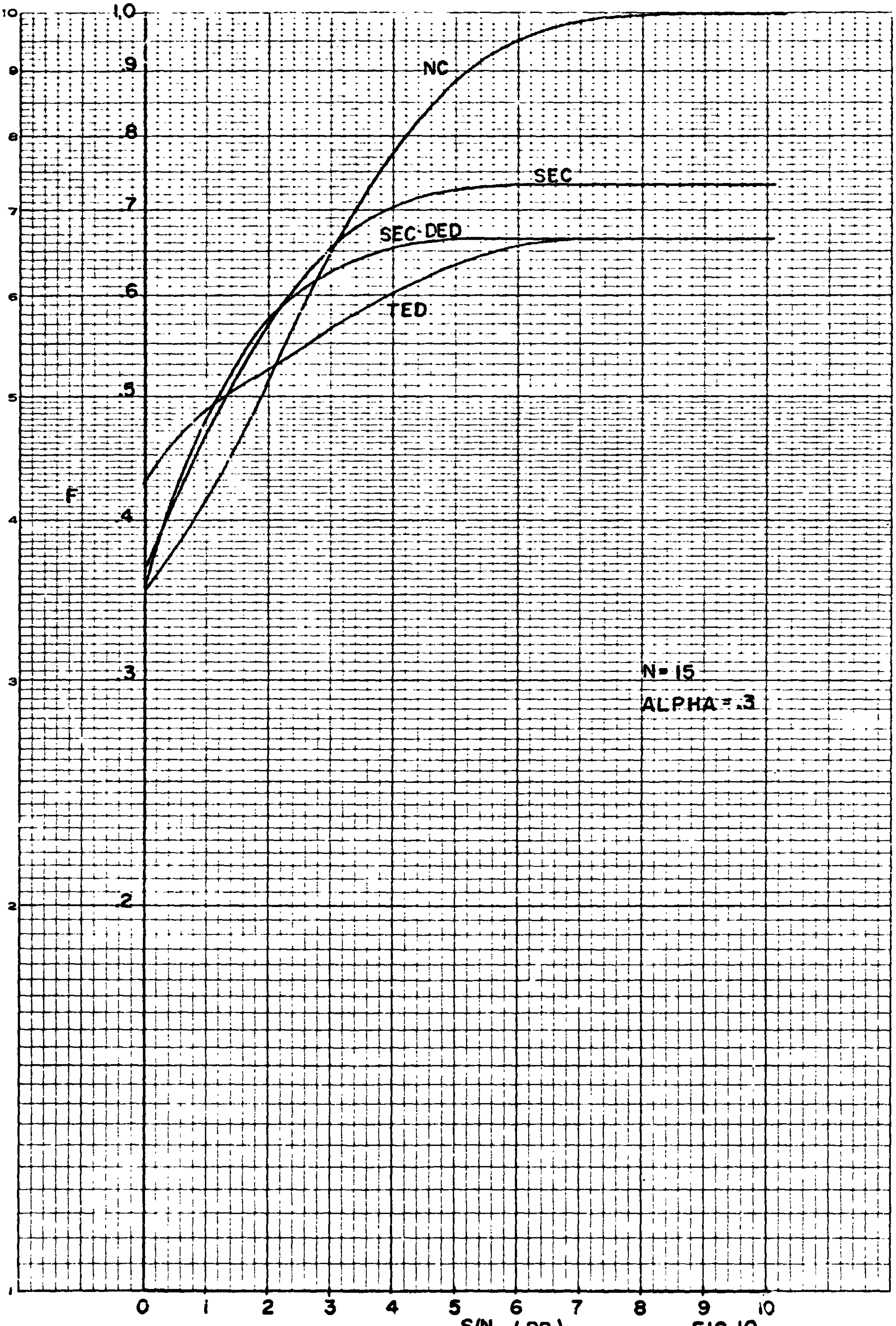


FIG. 10

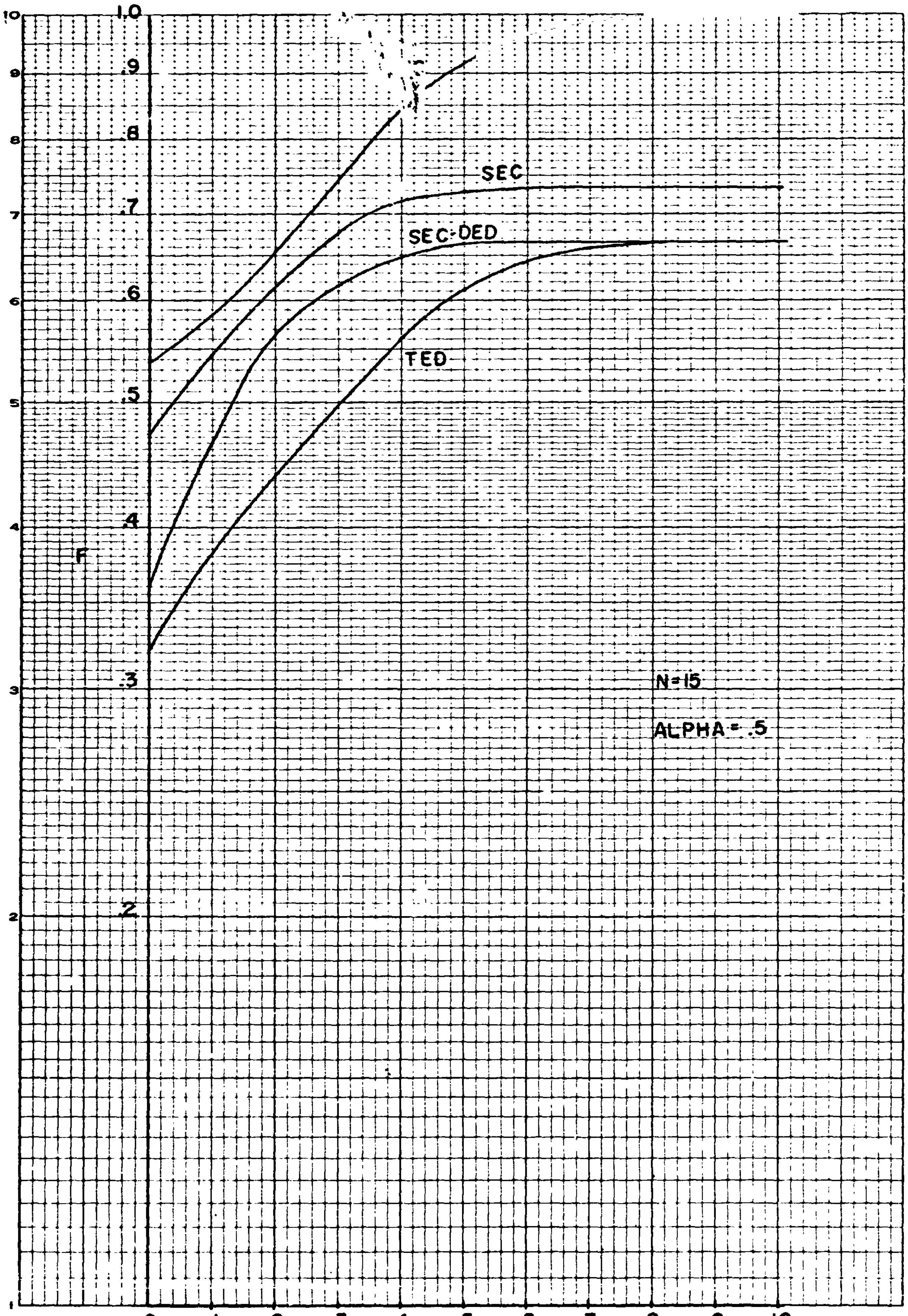
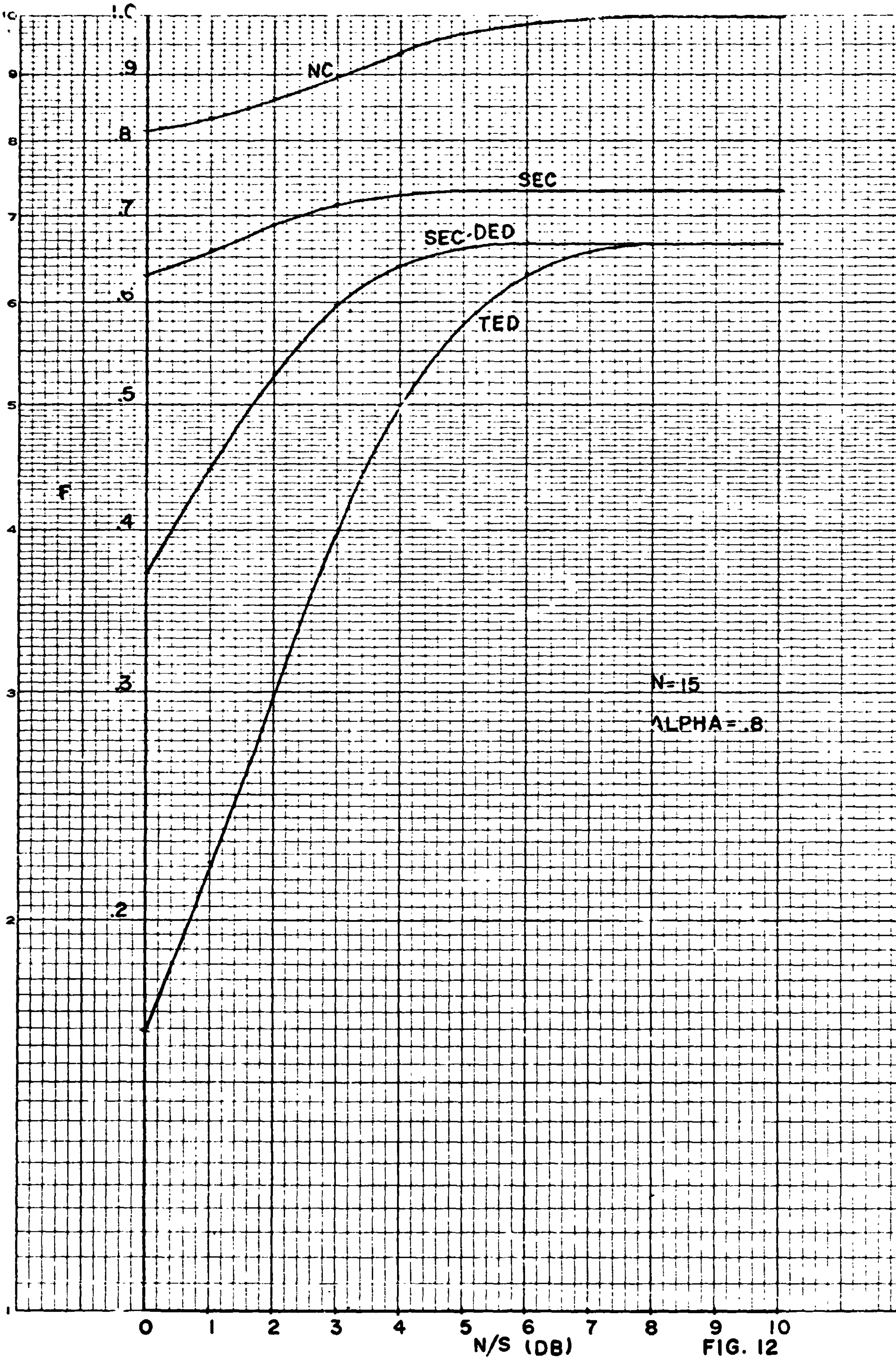


FIG. 11



NC
SEC
SEC-DED
TED
1.0
.9
.8
.7
.6
.5
.4
.3
.2
F
N=15
ALPHA=.8
0
1
2
3
4
5
6
7
8
9
10
N/S (DB)

FIG. 12

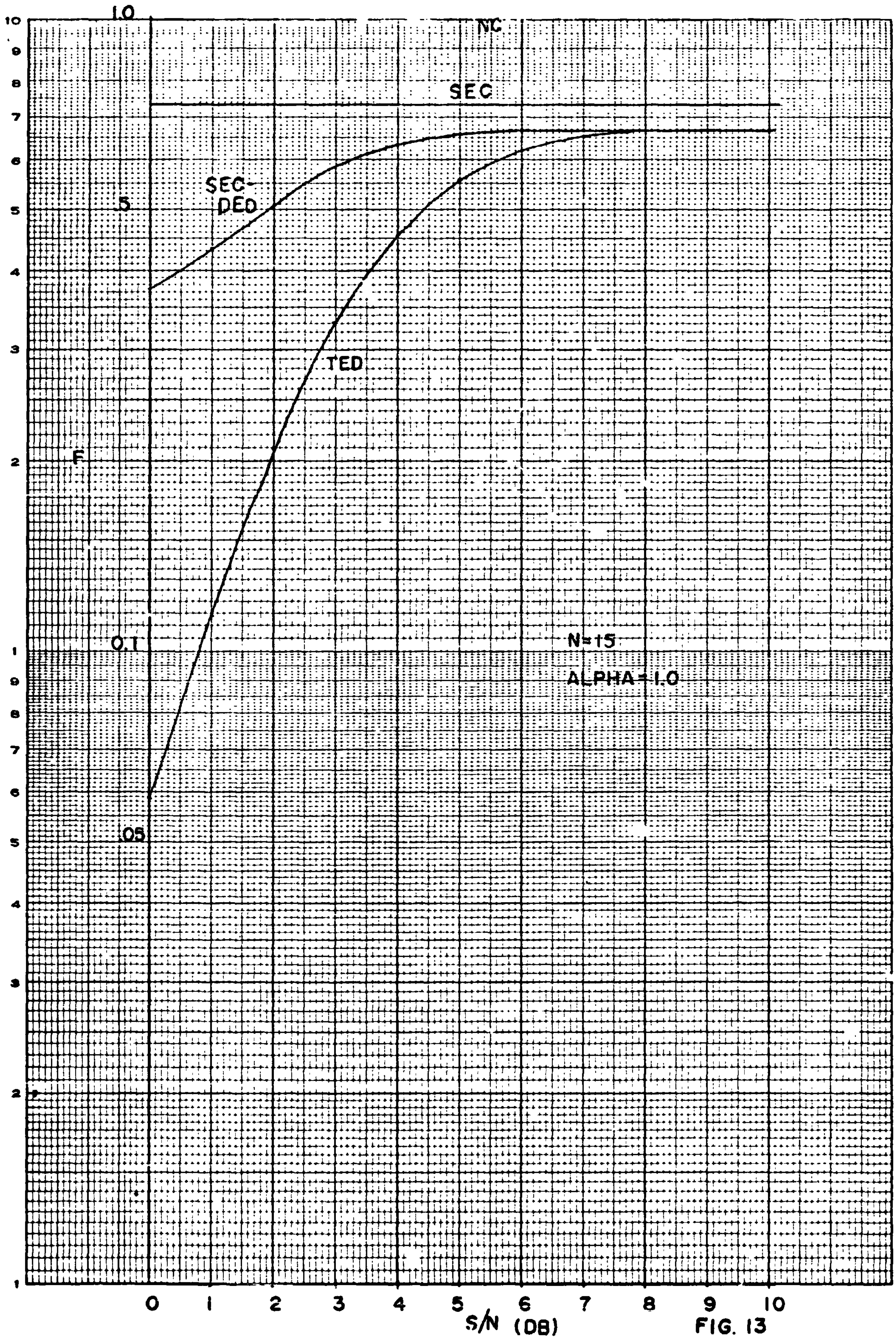


FIG. 13

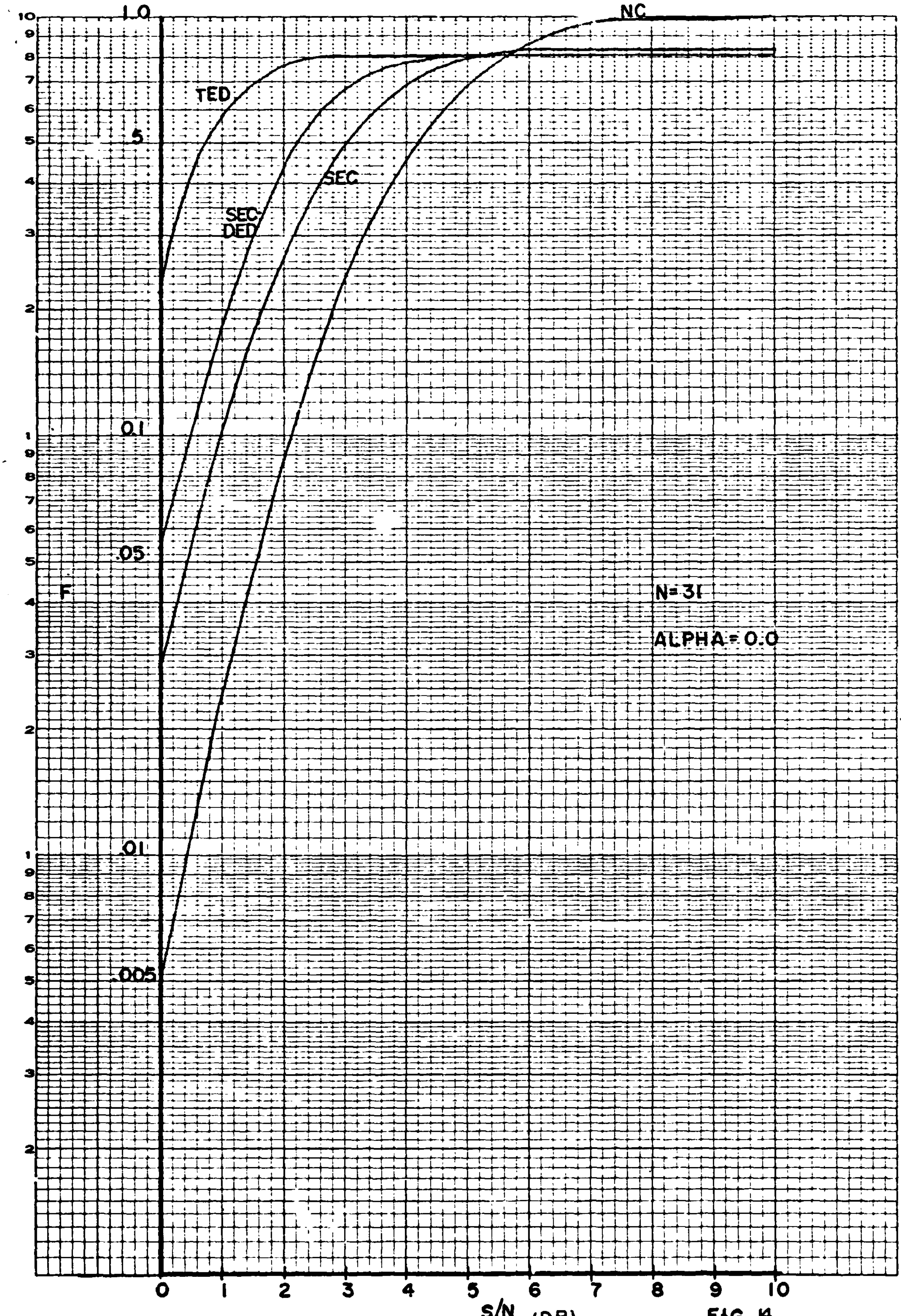
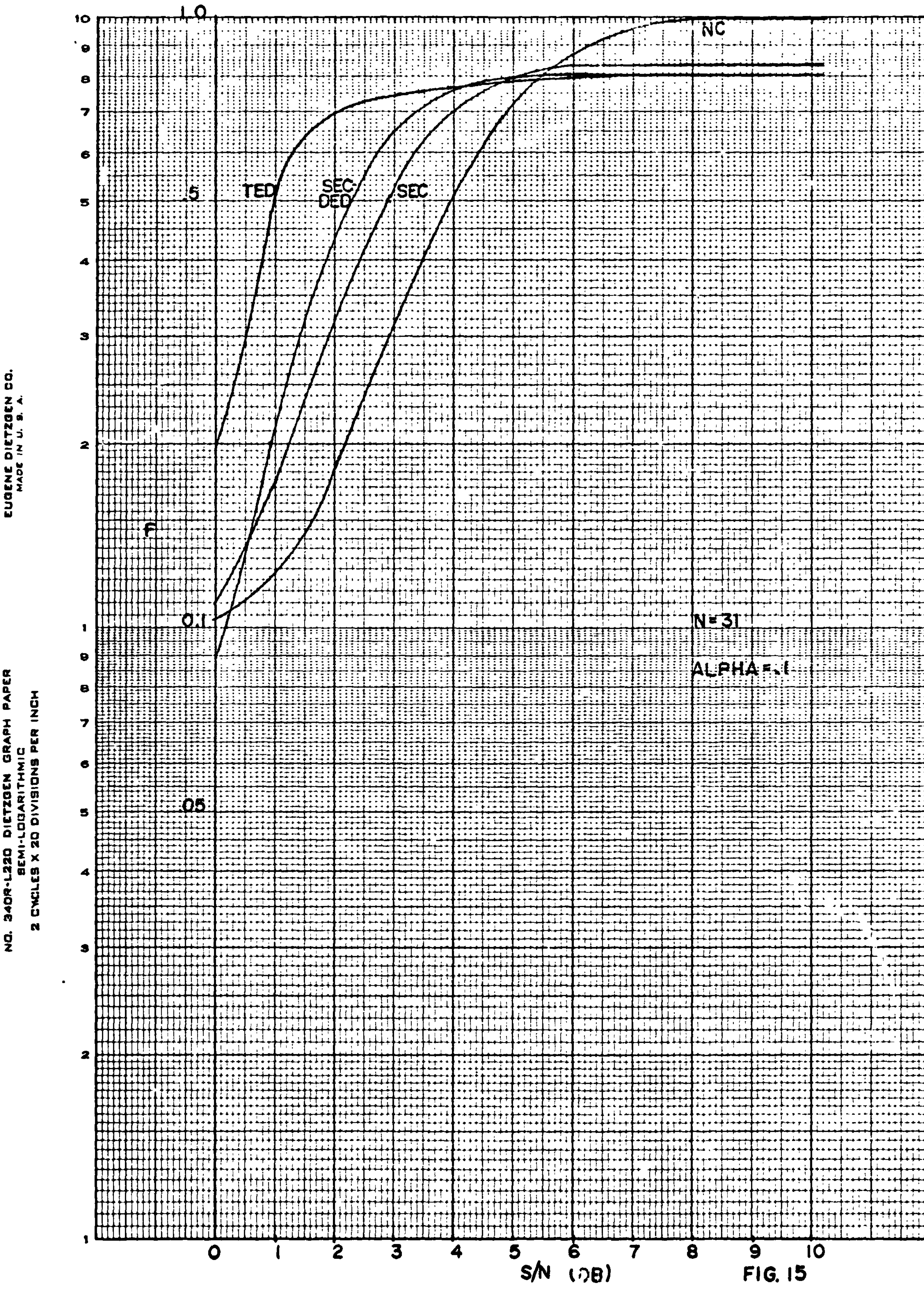
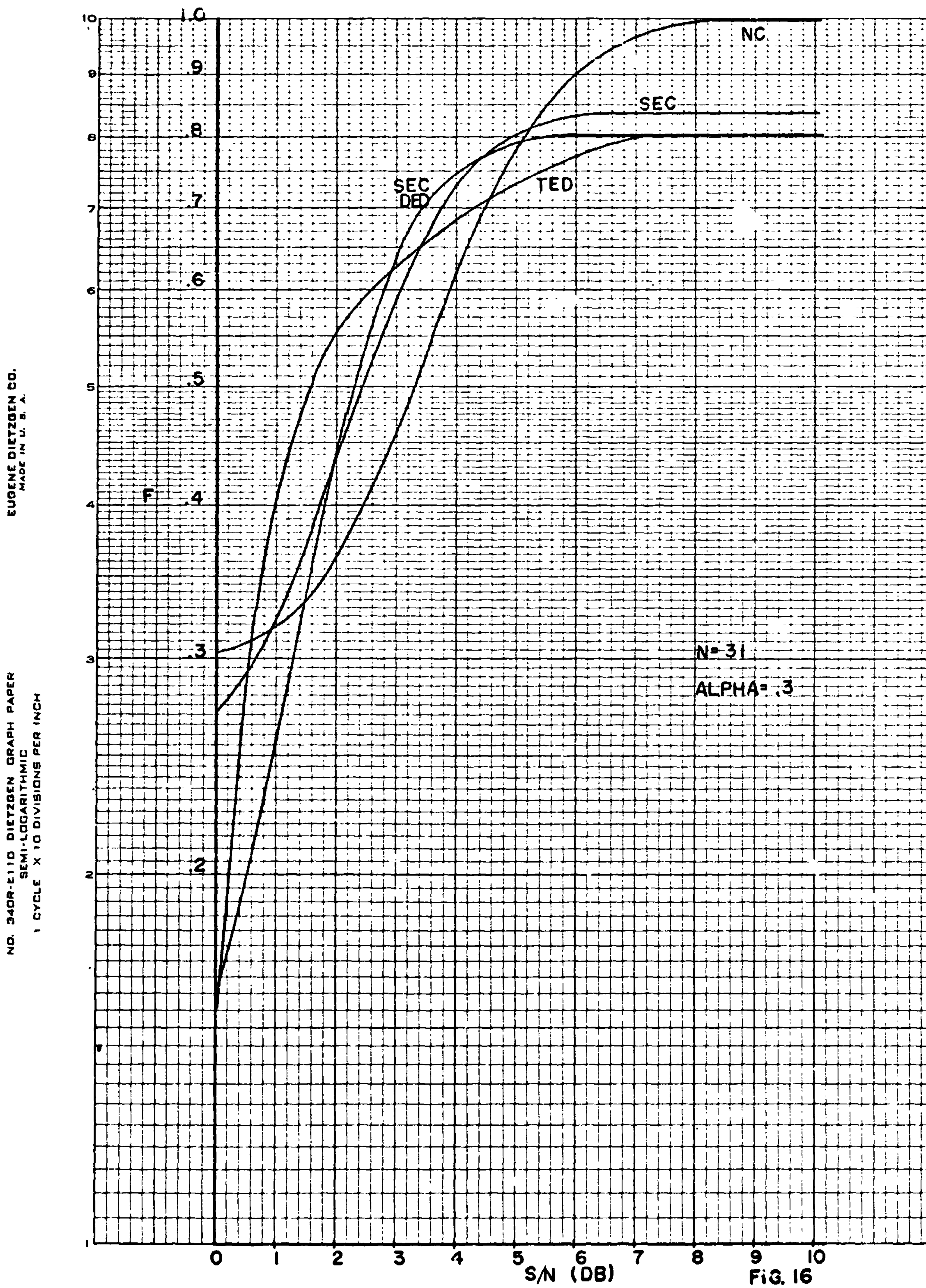

1.0
.5
0.1
.05
.01
.005
F
TED
SEC-DED
SEC
NC
N=31
ALPHA=0.0
0 1 2 3 4 5 6 7 8 9 10
S/N (DB)

FIG. 14



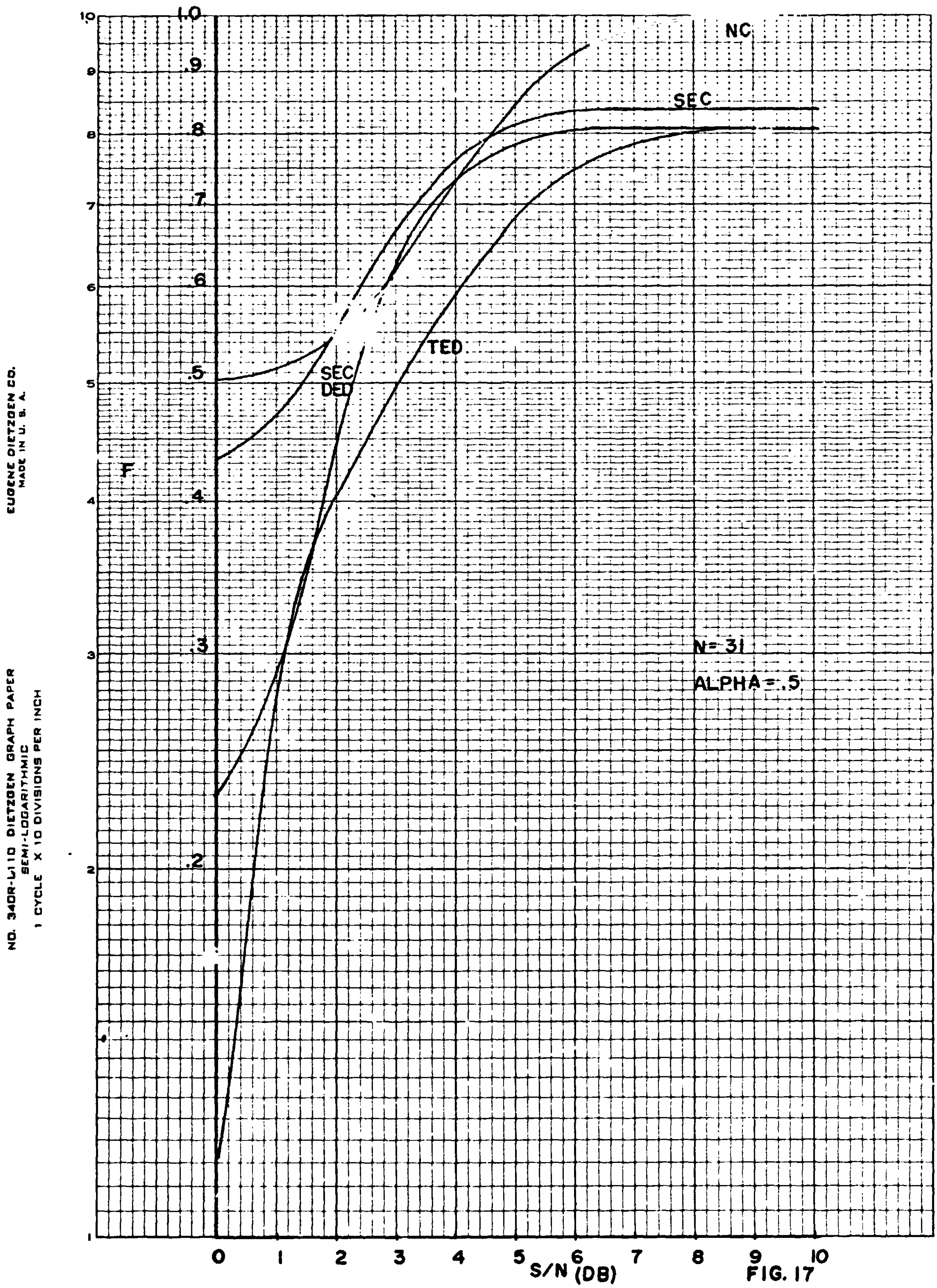
1.0
.5
0.1
.05
F
TED
SEC
DED
SEC
NC
N=31
ALPHA=.1
0
1
2
3
4
5
6
7
8
9
10
S/N (DB)
EUGENE DIETZGEN CO.
MADE IN U. S. A.
NO. 340R-L220 DIETZGEN GRAPH PAPER
SEMI-LOGARITHMIC
2 CYCLES X 20 DIVISIONS PER INCH

FIG. 15



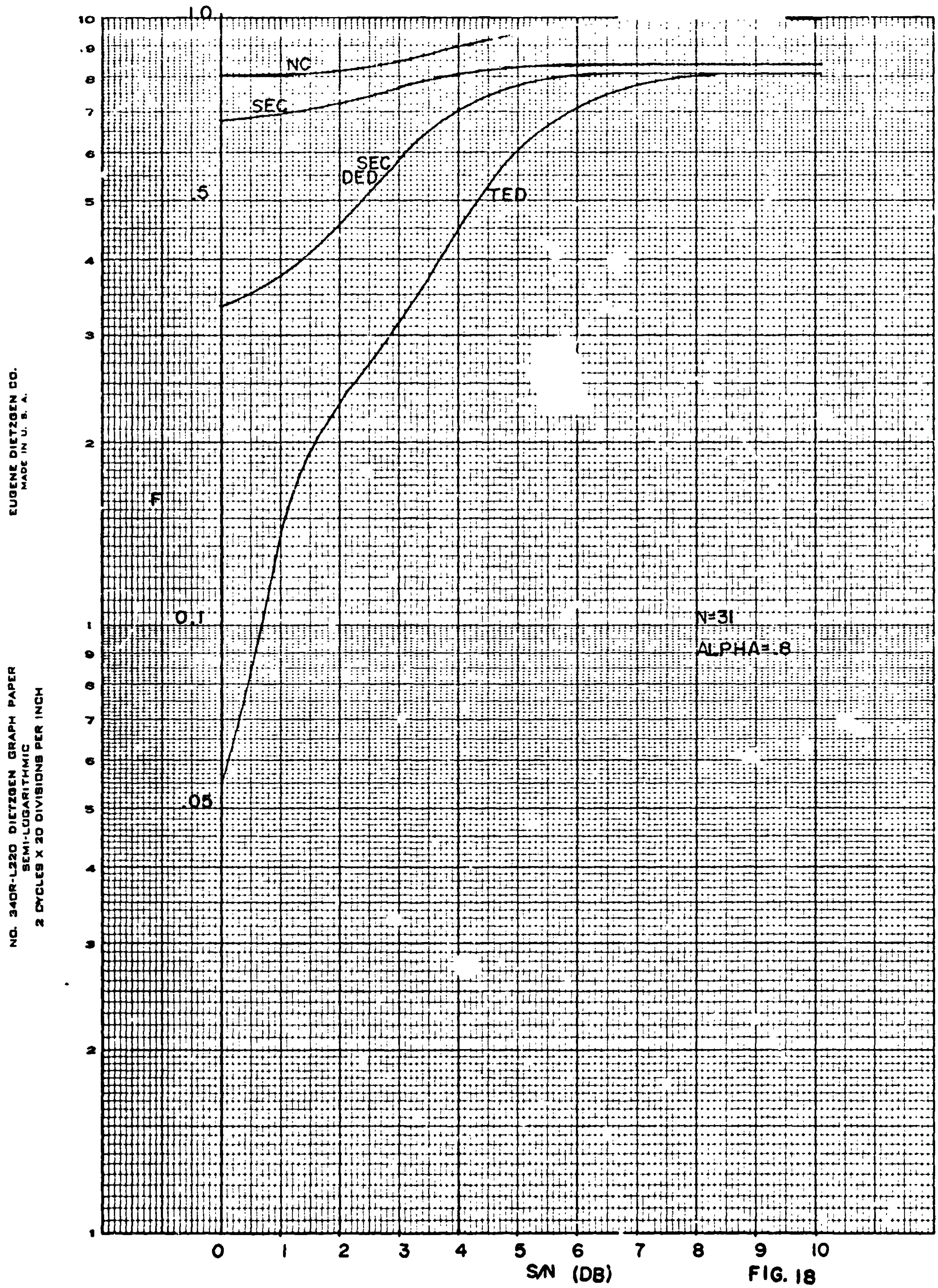
NC
SEC
SEC
DED
TED
F
1.0
.9
.8
.7
.6
.5
.4
.3
.2
N= 31
ALPHA= .3
0
1
2
3
4
5
6
7
8
9
10
S/N (DB)
EUGENE DIETZGEN CO.
MADE IN U. S. A.
NO. 340R-E110 DIETZGEN GRAPH PAPER
SEMI-LOGARITHMIC
1 CYCLE X 10 DIVISIONS PER INCH

FIG. 16



NC
SEC
SEC
DED
TED
F
N= 31
ALPHA = .5
S/N (DB)
NO. 340R-L110 DIETZGEN GRAPH PAPER
SEMI-LOGARITHMIC
1 CYCLE X 10 DIVISIONS PER INCH
EUGENE DIETZGEN CO.
MADE IN U. S. A.

FIG. 17



1.0
.5
0.1
.05
F
NC
SEC
SEC
DED
TED
N=31
ALPHA=.8
0
1
2
3
4
5
6
7
8
9
10
S/N (DB)
EUGENE DIETZGEN CO.
MADE IN U. S. A.
NO. 340R-L220 DIETZGEN GRAPH PAPER
SEMI-LOGARITHMIC
2 CYCLES X 20 DIVISIONS PER INCH

FIG. 18

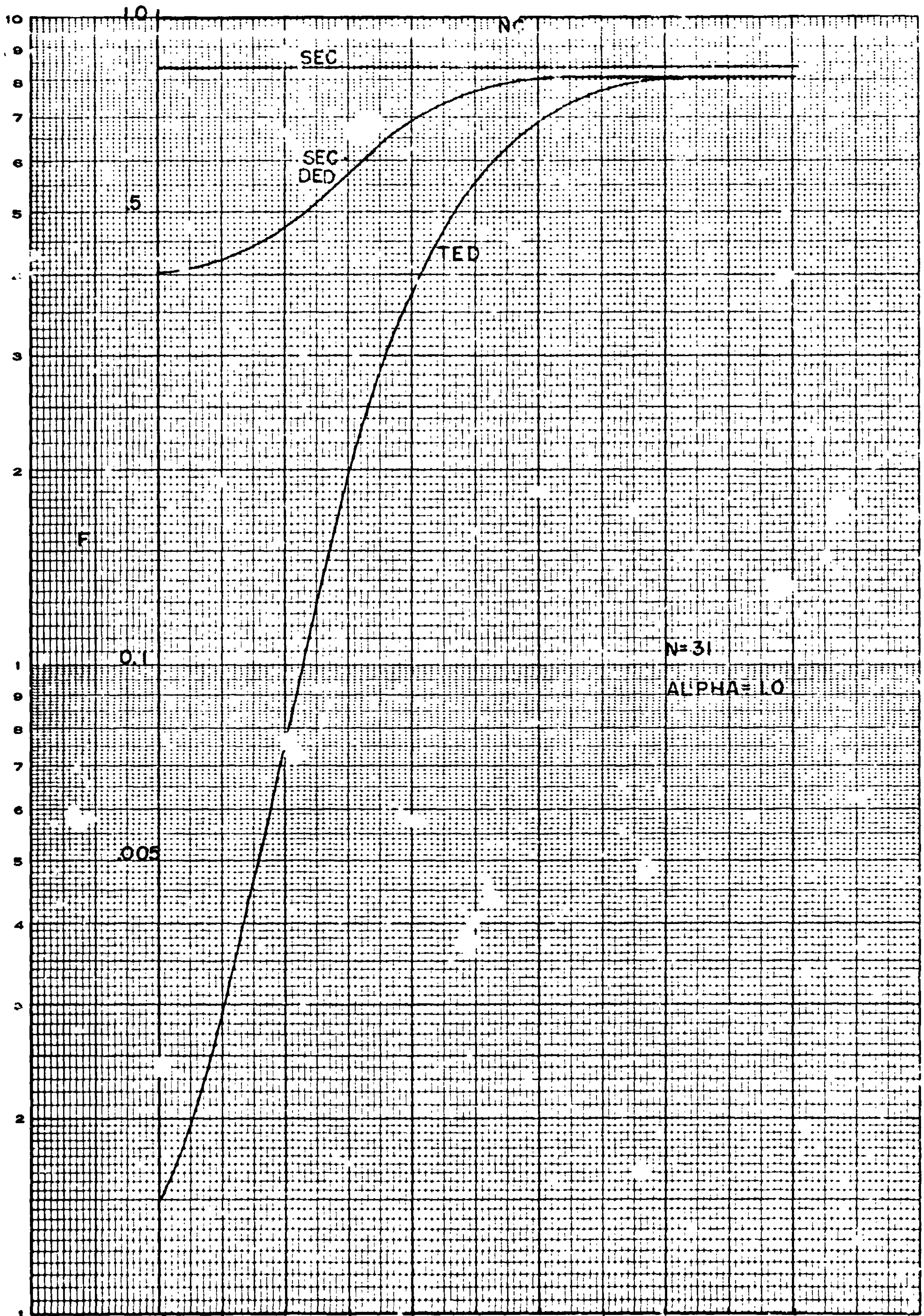
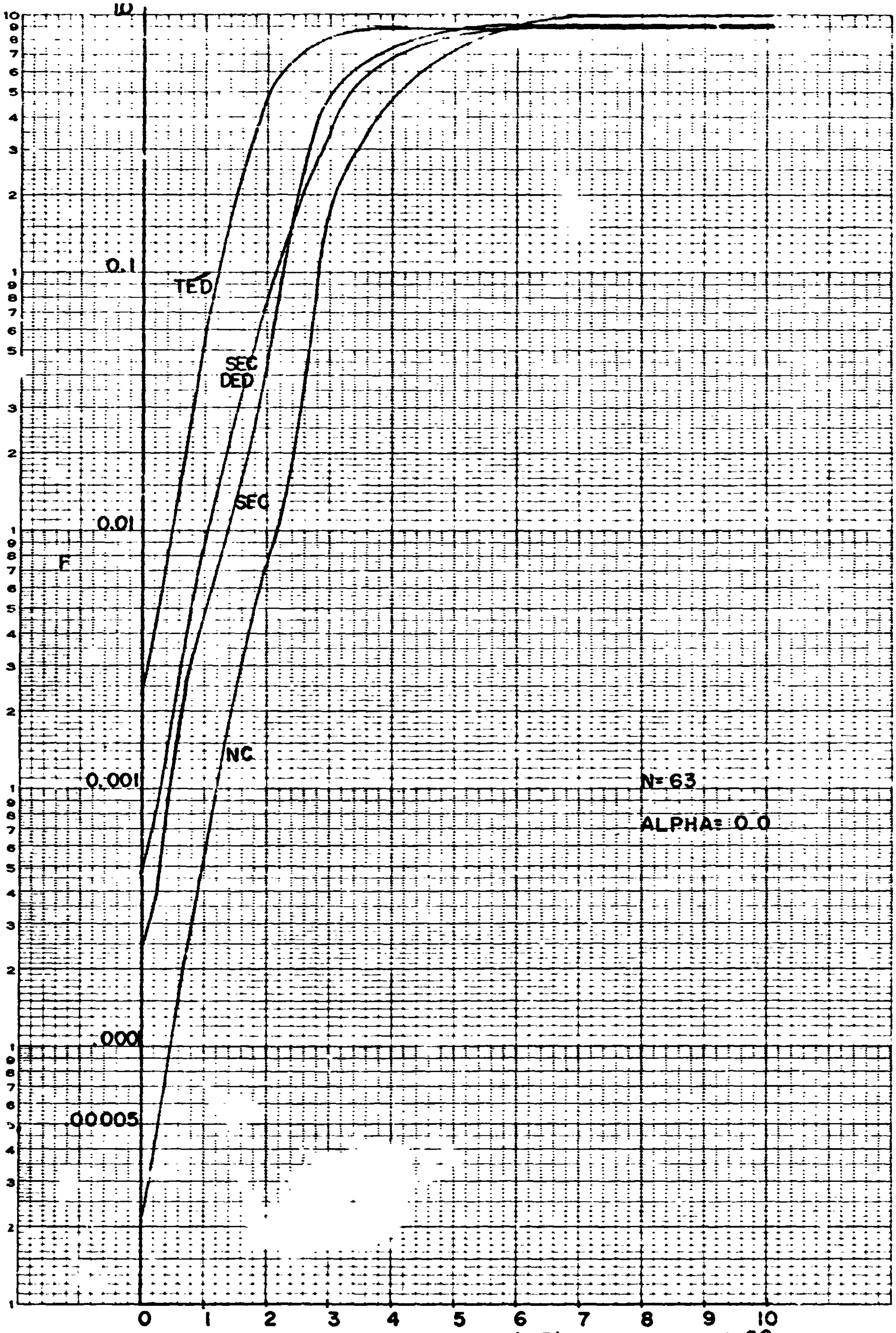
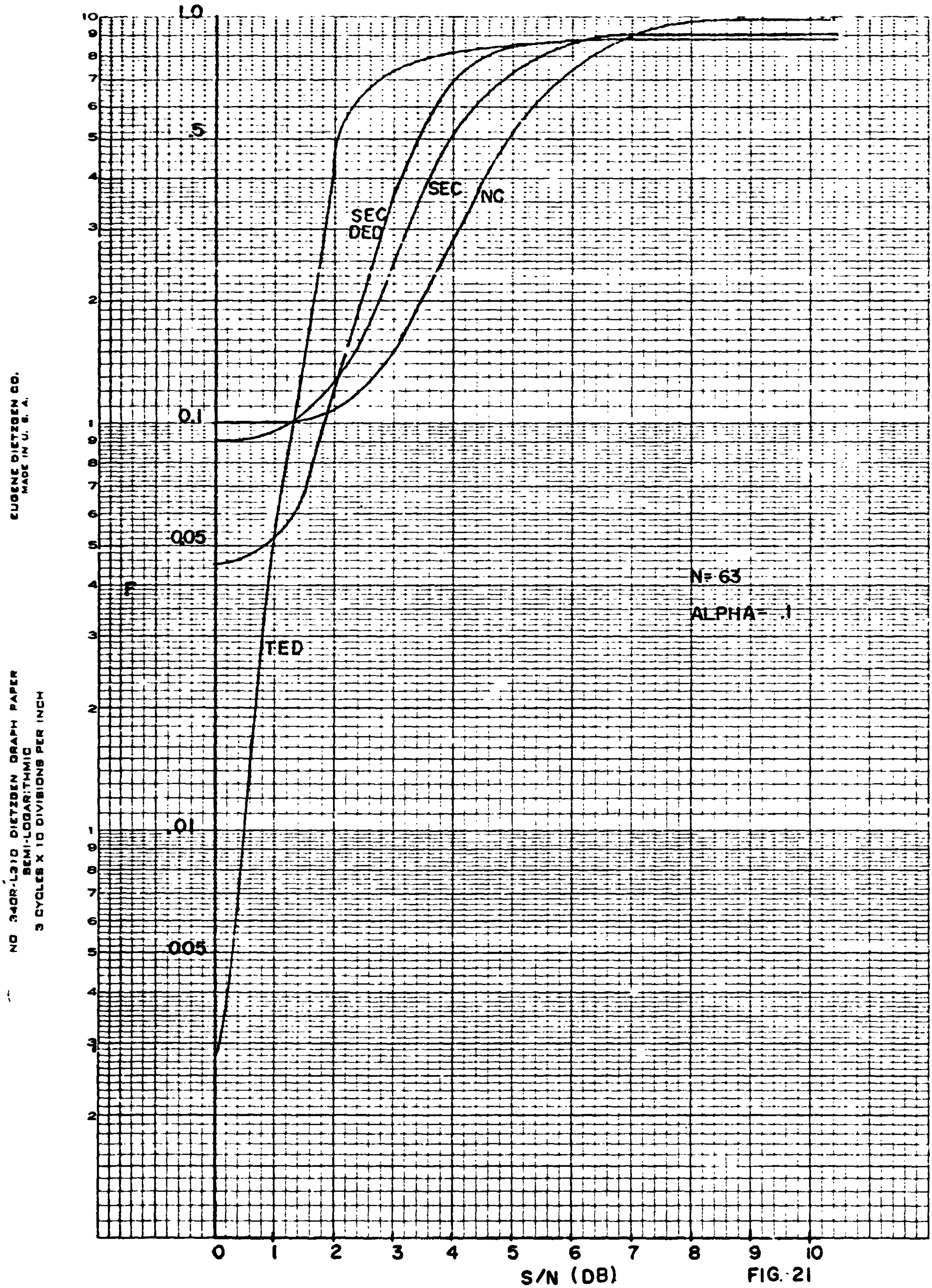


FIG. 19



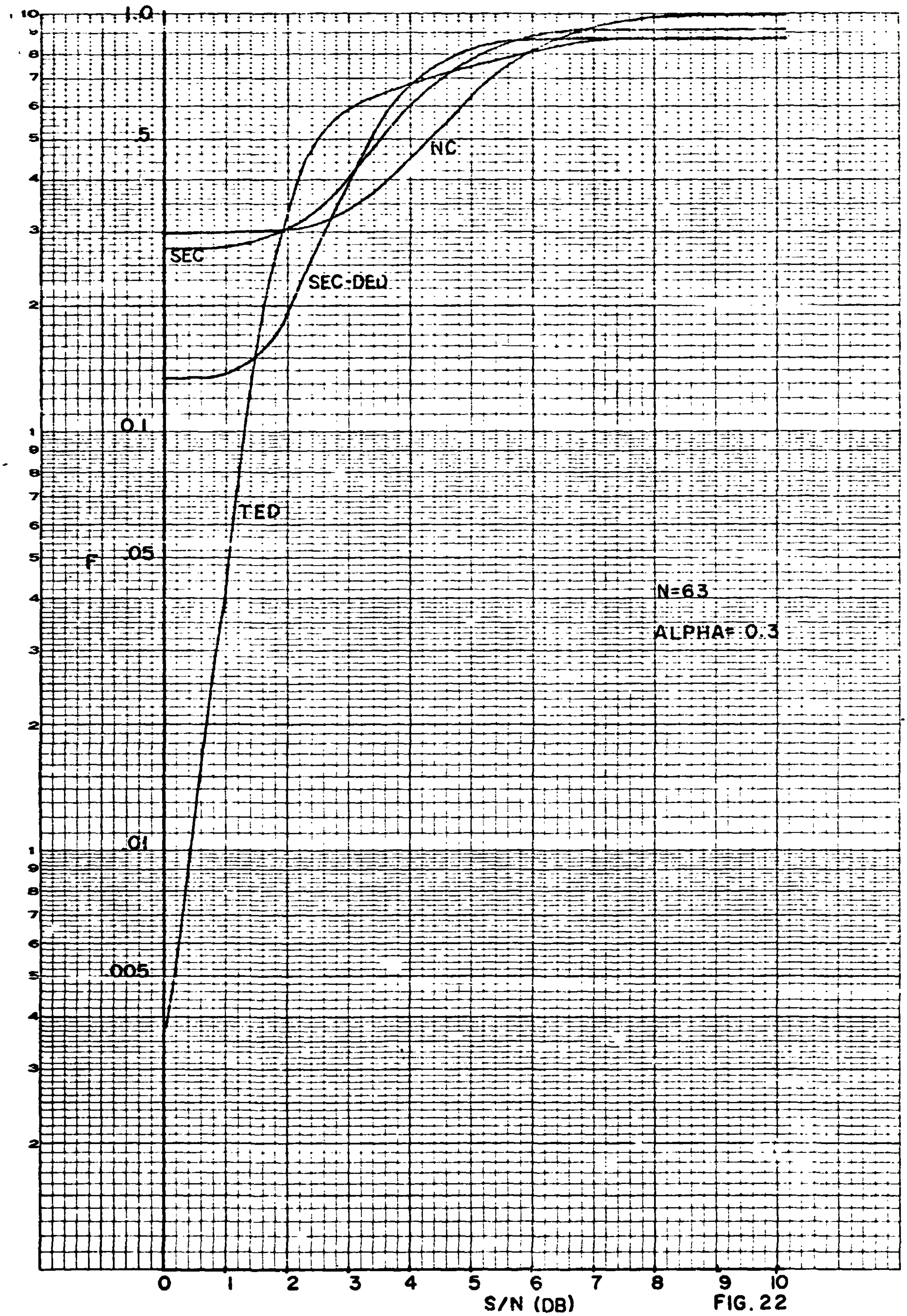
10
0.1
0.01
0.001
.0001
.00005
F
TED
SEC
DED
SEC
NC
N=63
ALPHA= 0.0
0 1 2 3 4 5 6 7 8 9 10
S/N (DB)

FIG. 20



1.0
.5
SEC
DED
SEC
NC
0.1
.005
.05
.01
N= 63
ALPHA= .1
TED
F
0
1
2
3
4
5
6
7
8
9
10
S/N (DB)
EUGENE DIETZGEN CO.
MADE IN U. S. A.
NO. 340R-L310 DIETZGEN GRAPH PAPER
SEMI-LOGARITHMIC
3 CYCLES X 10 DIVISIONS PER INCH

FIG. 21


1.0
.5
0.1
.05
.01
.005
F
SEC
SEC-DED
NC
TED
N=63
ALPHA= 0.3
0
1
2
3
4
5
6
7
8
9
10
S/N (DB)
FIG. 22

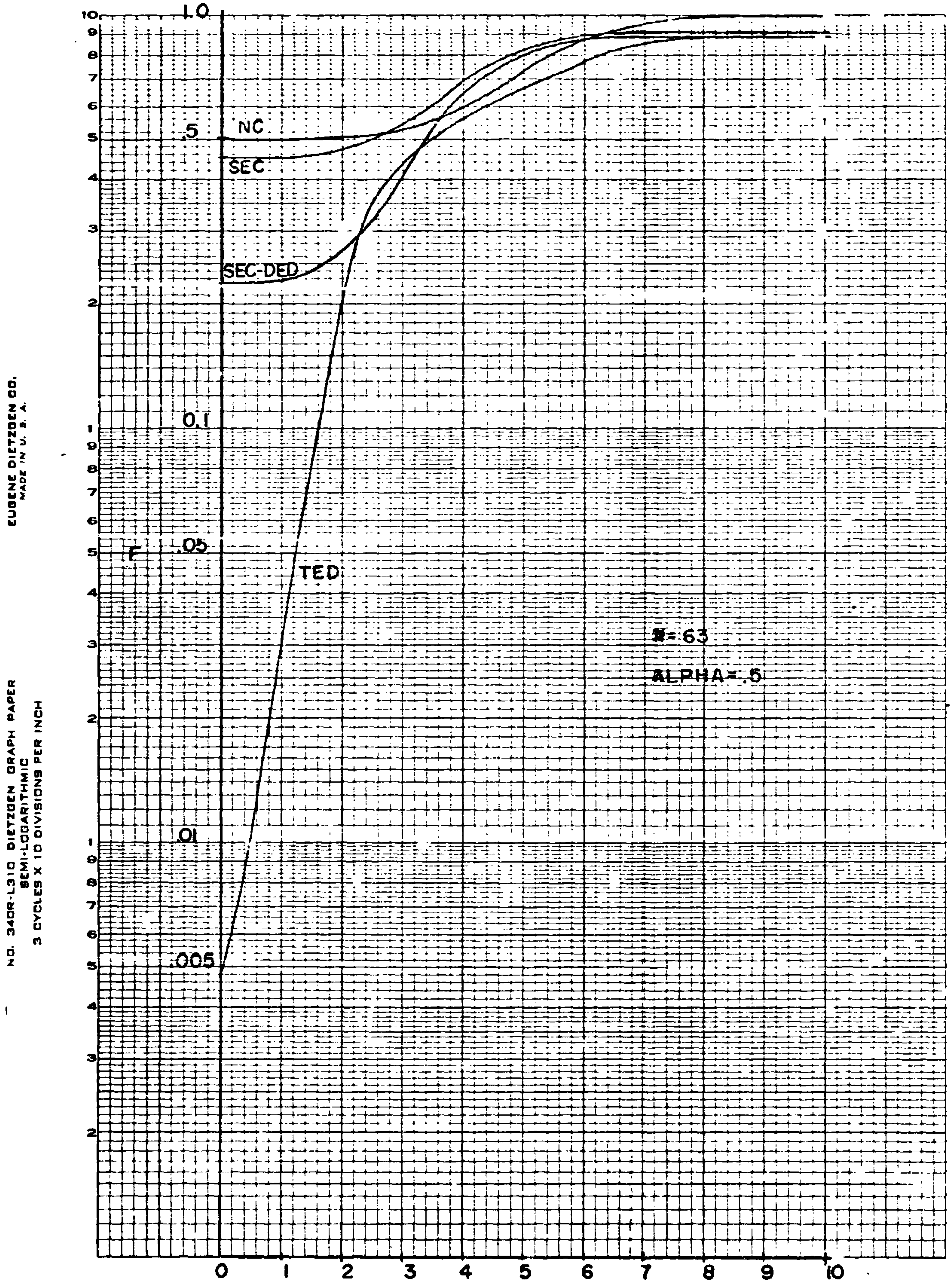

1.0
.5
0.1
.05
.01
.005
F
NC
SEC
SEC-DED
TED
N = 63
ALPHA = .5
0
1
2
3
4
5
6
7
8
9
10
S/N (DB)
EUGENE DIETZGEN CO.
MADE IN U. S. A.
NO. 340R-L310 DIETZGEN GRAPH PAPER
SEMI-LOGARITHMIC
3 CYCLES X 10 DIVISIONS PER INCH

FIG. 23

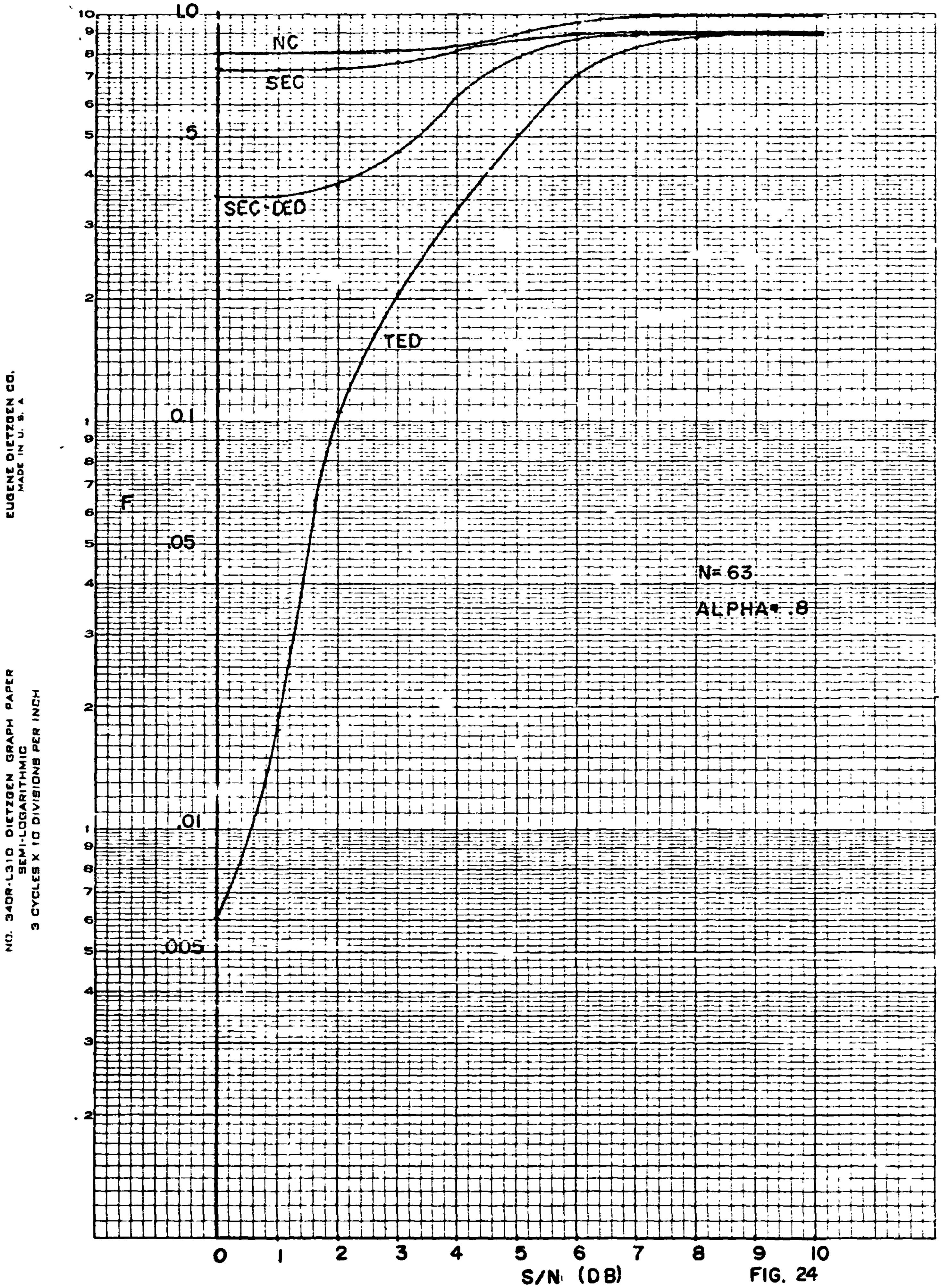
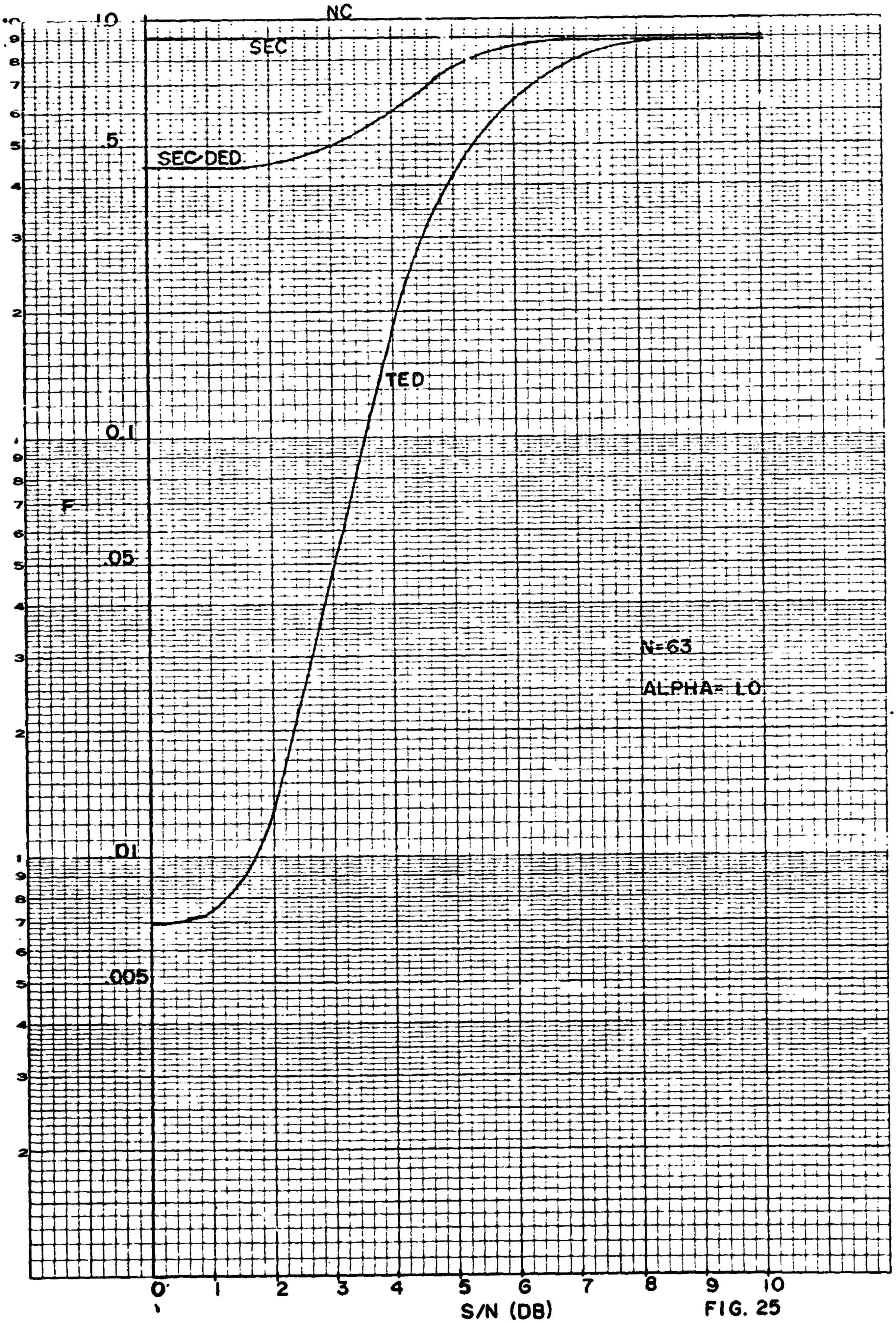
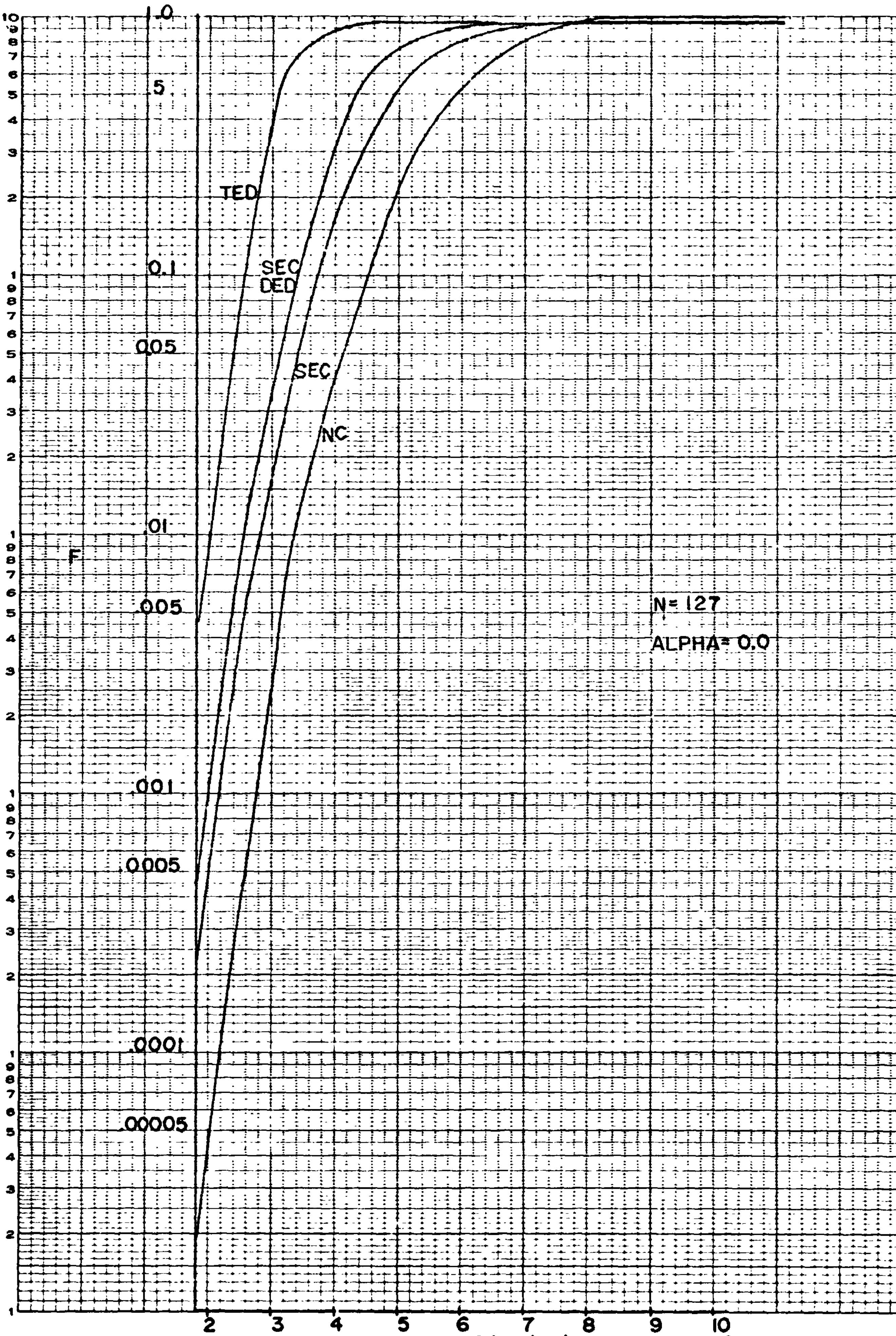

EUGENE DIETZGEN CO.
MADE IN U. S. A.
NO. 340R-L310 DIETZGEN GRAPH PAPER
SEMI-LOGARITHMIC
3 CYCLES X 10 DIVISIONS PER INCH
1.0
.5
.1
.05
.01
.005
F
NC
SEC
SEC-DED
TED
N= 63
ALPHA= .8
0
1
2
3
4
5
6
7
8
9
10
S/N (DB)

FIG. 24



1.0
.5
0.1
.05
.01
.005
F
NC
SEC
SEC-DED
TED
N=63
ALPHA=1.0
0
1
2
3
4
5
6
7
8
9
10
S/N (DB)

FIG. 25



1.0
.5
.1
.05
.01
.005
.001
.0005
.0001
.00005
F
TED
SEC
DED
SEC
NC
N= 127
ALPHA= 0.0
2
3
4
5
6
7
8
9
10
S/N (DB)

FIG. 26

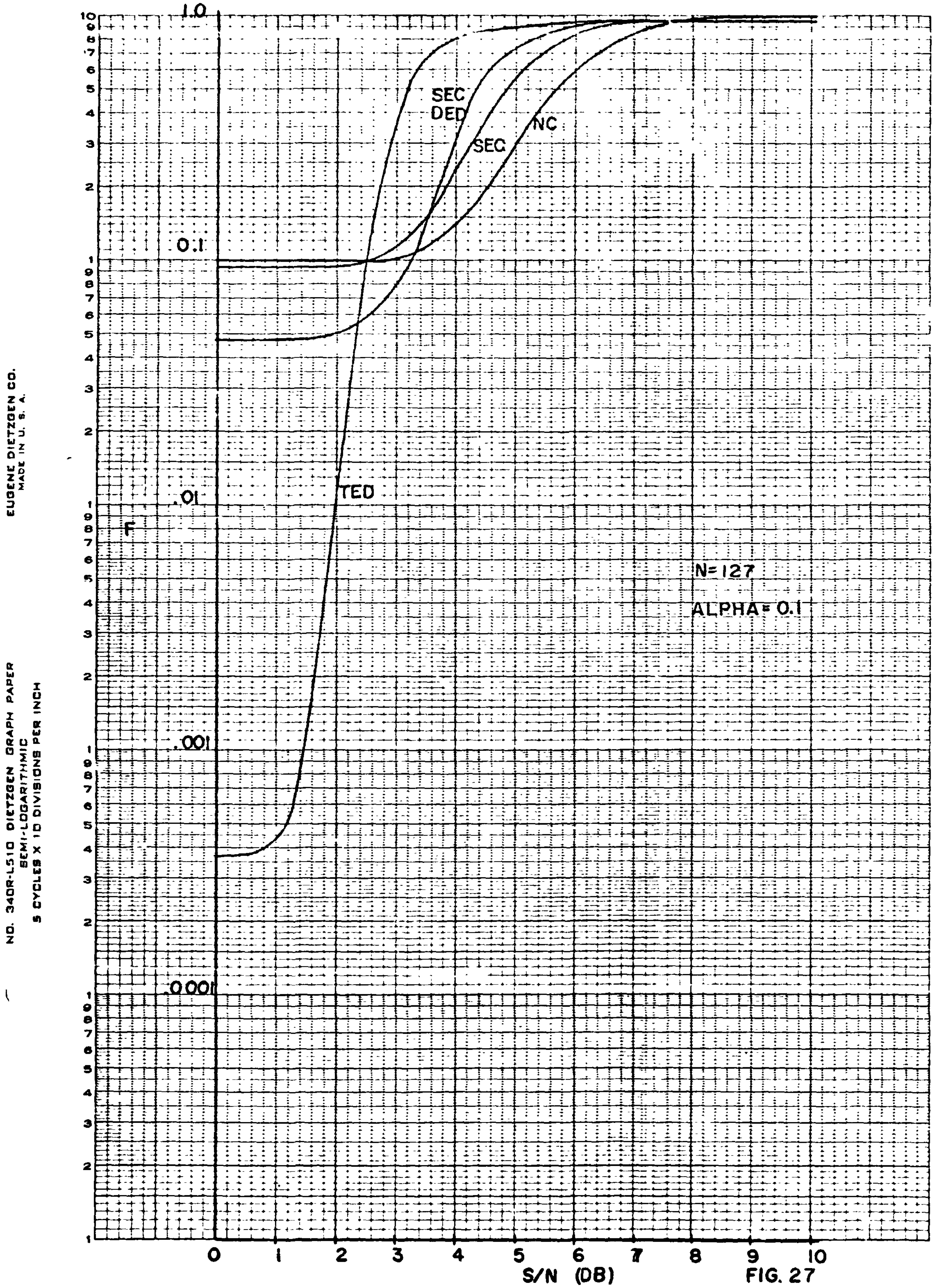
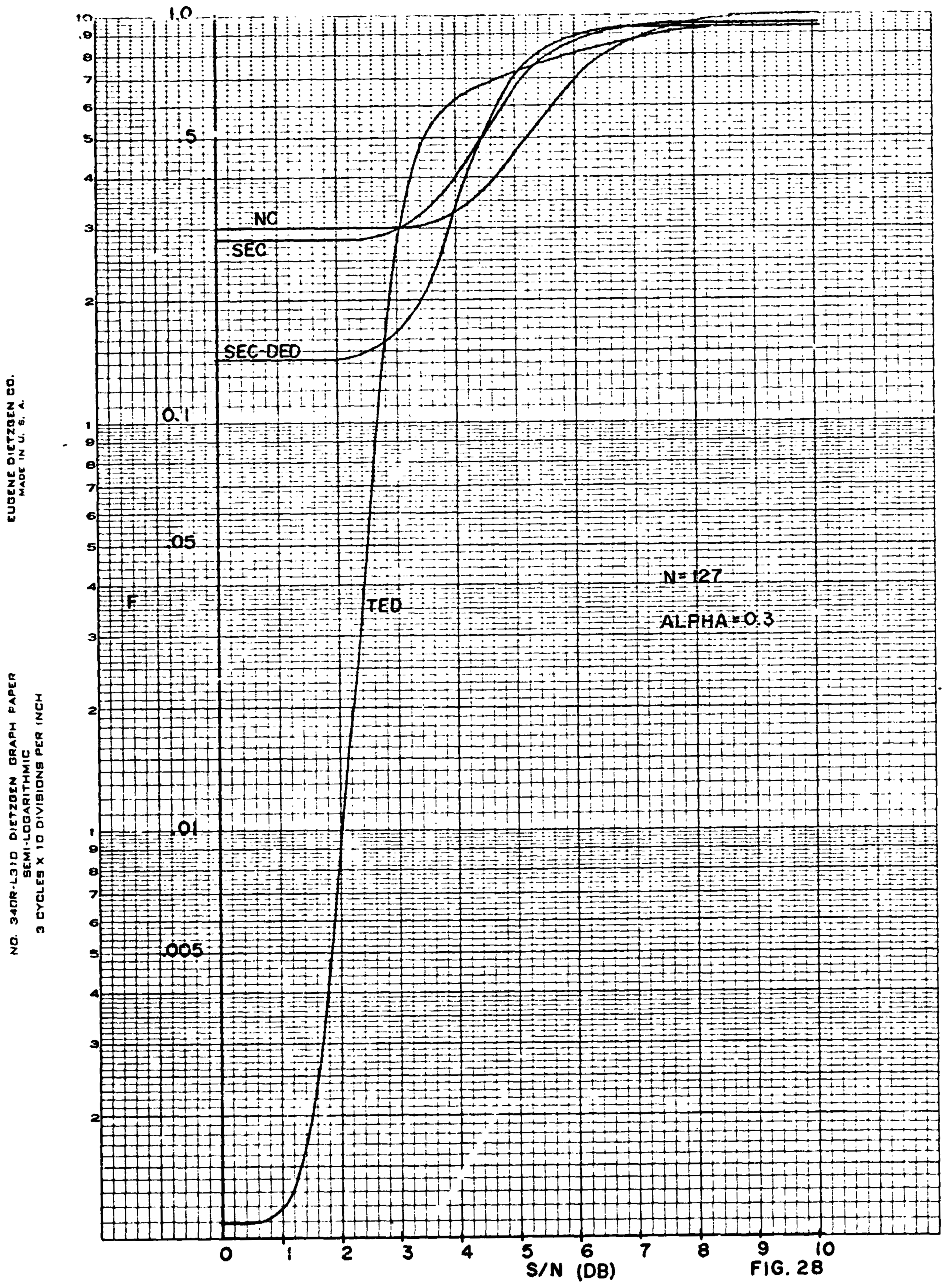

1.0
0.1
.01
.001
.0001
F
SEC
DED
SEC
NC
TED
N=127
ALPHA= 0.1
0
1
2
3
4
5
6
7
8
9
10
S/N (DB)
EUGENE DIETZGEN CO.
MADE IN U. S. A.
NO. 340R-L510 DIETZGEN GRAPH PAPER
SEMI-LOGARITHMIC
5 CYCLES X 10 DIVISIONS PER INCH

FIG. 27



EUGENE DIETZGEN CO.
MADE IN U. S. A.
NO. 340R-L310 DIETZGEN GRAPH PAPER
SEMI-LOGARITHMIC
3 CYCLES X 10 DIVISIONS PER INCH
1.0
.5
0.1
.05
.01
.005
F
NC
SEC
SEC-DED
TED
N=127
ALPHA=0.3
0
1
2
3
4
5
6
7
8
9
10
S/N (DB)

FIG. 28

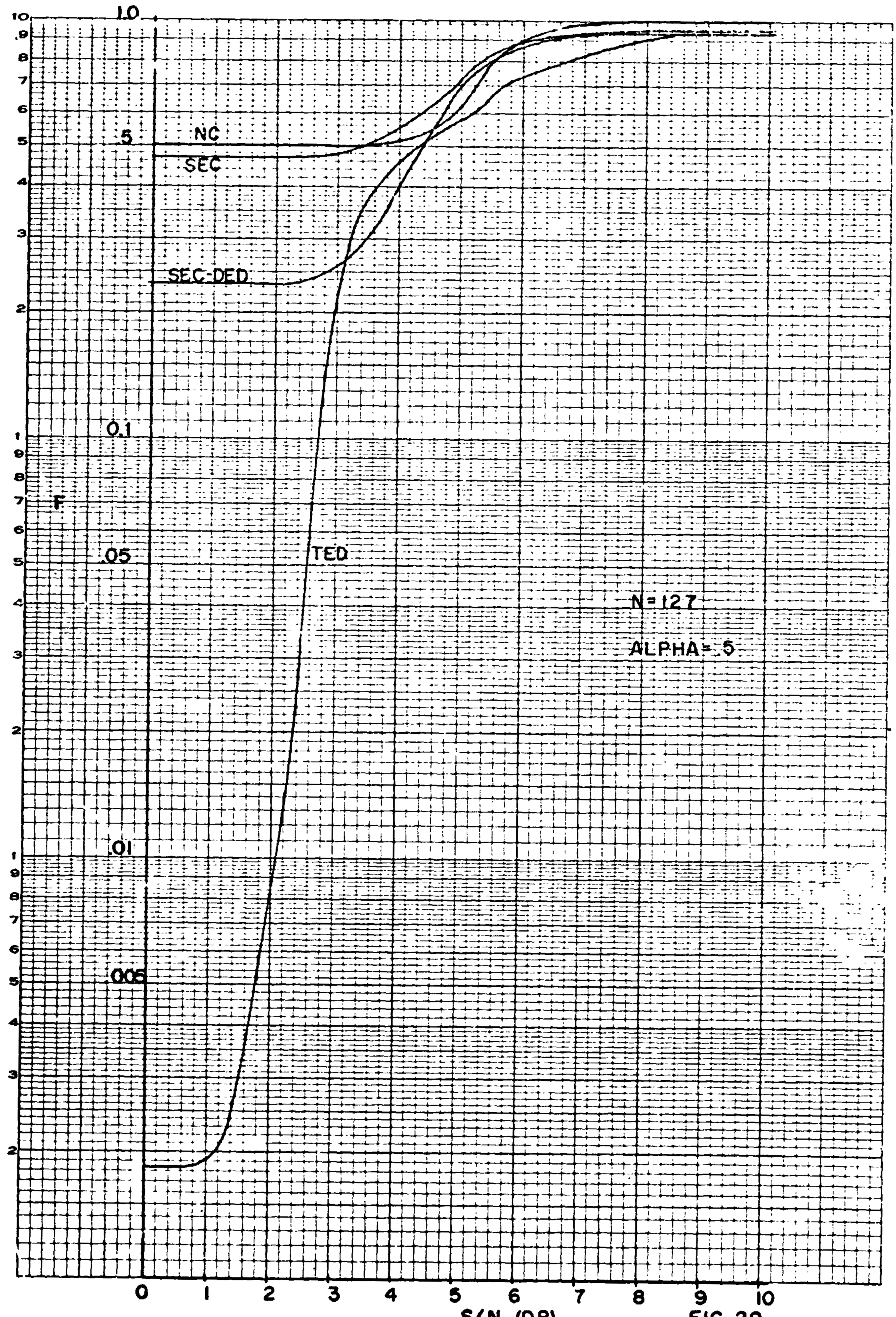
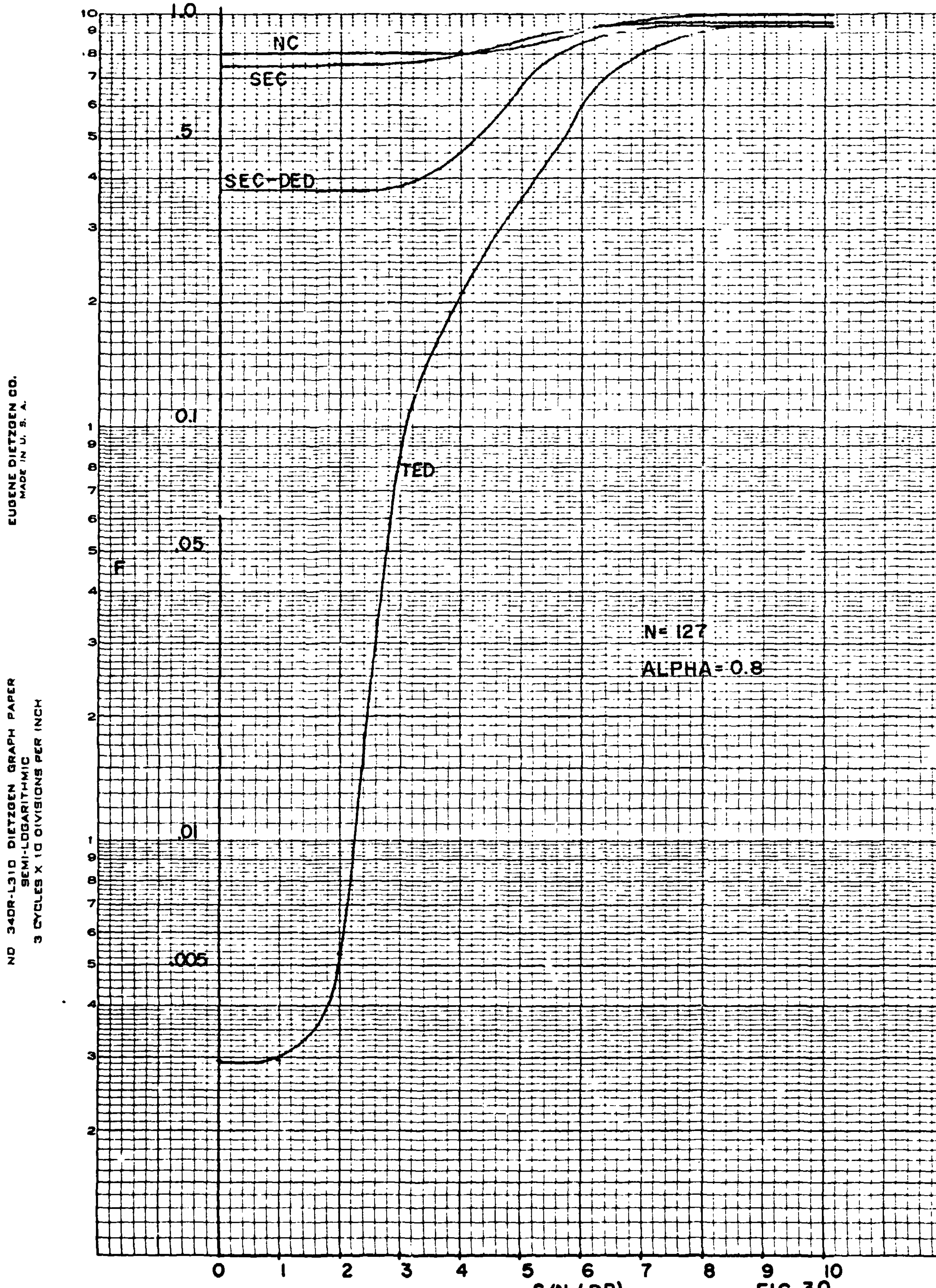

1.0
.5
0.1
.05
.01
.005
F
NC
SEC
SEC-DED
TED
N=127
ALPHA=.5
0
1
2
3
4
5
6
7
8
9
10
S/N (DB)

FIG. 29



1.0
.5
0.1
.05
.01
.005
F
NC
SEC
SEC-DED
TED
N= 127
ALPHA= 0.8
0
1
2
3
4
5
6
7
8
9
10
S/N (DB)
EUGENE DIETZGEN CO.
MADE IN U. S. A.
NO 340R-L310 DIETZGEN GRAPH PAPER
SEMI-LOGARITHMIC
3 CYCLES X 10 DIVISIONS PER INCH

FIG. 30

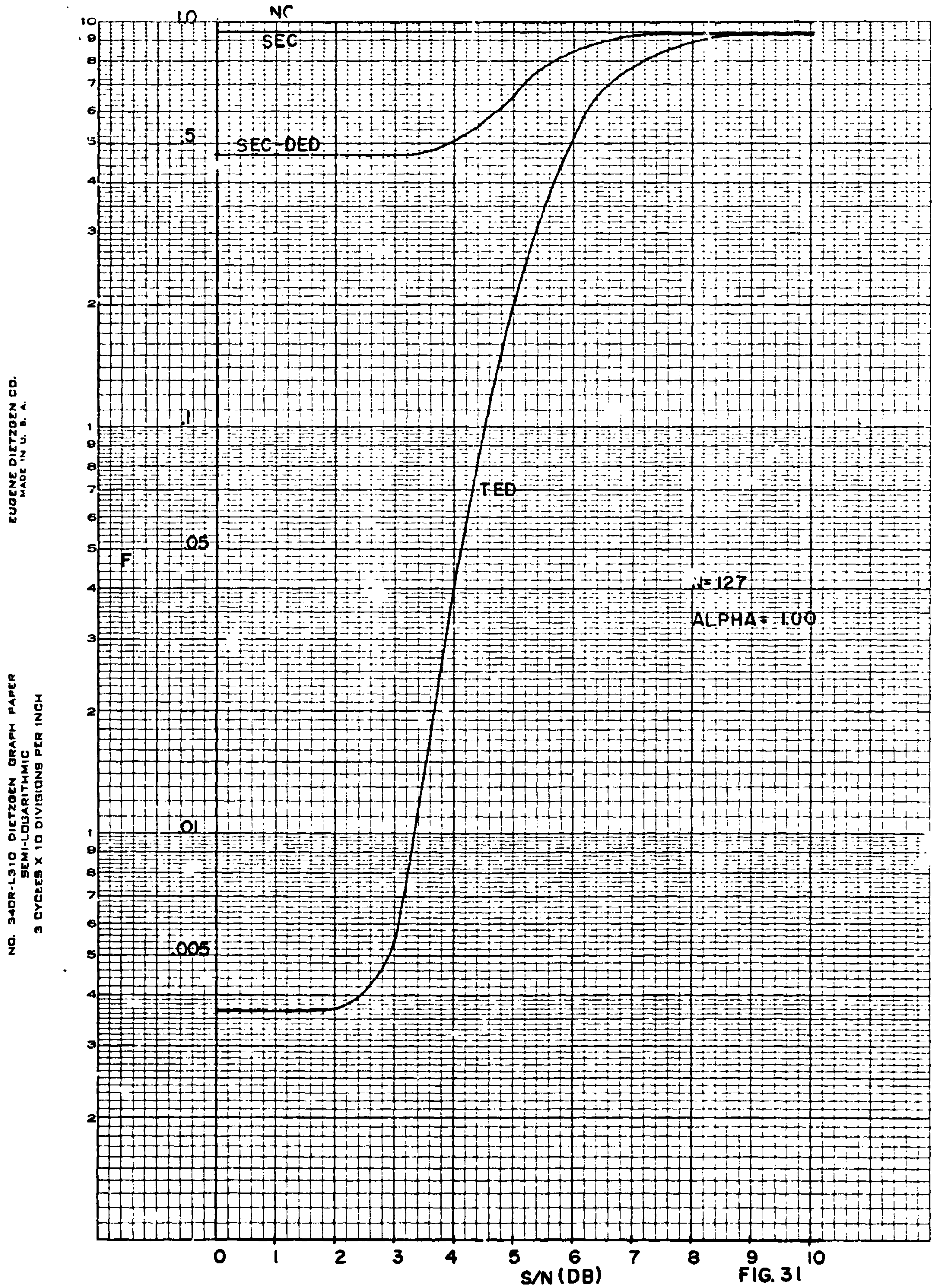
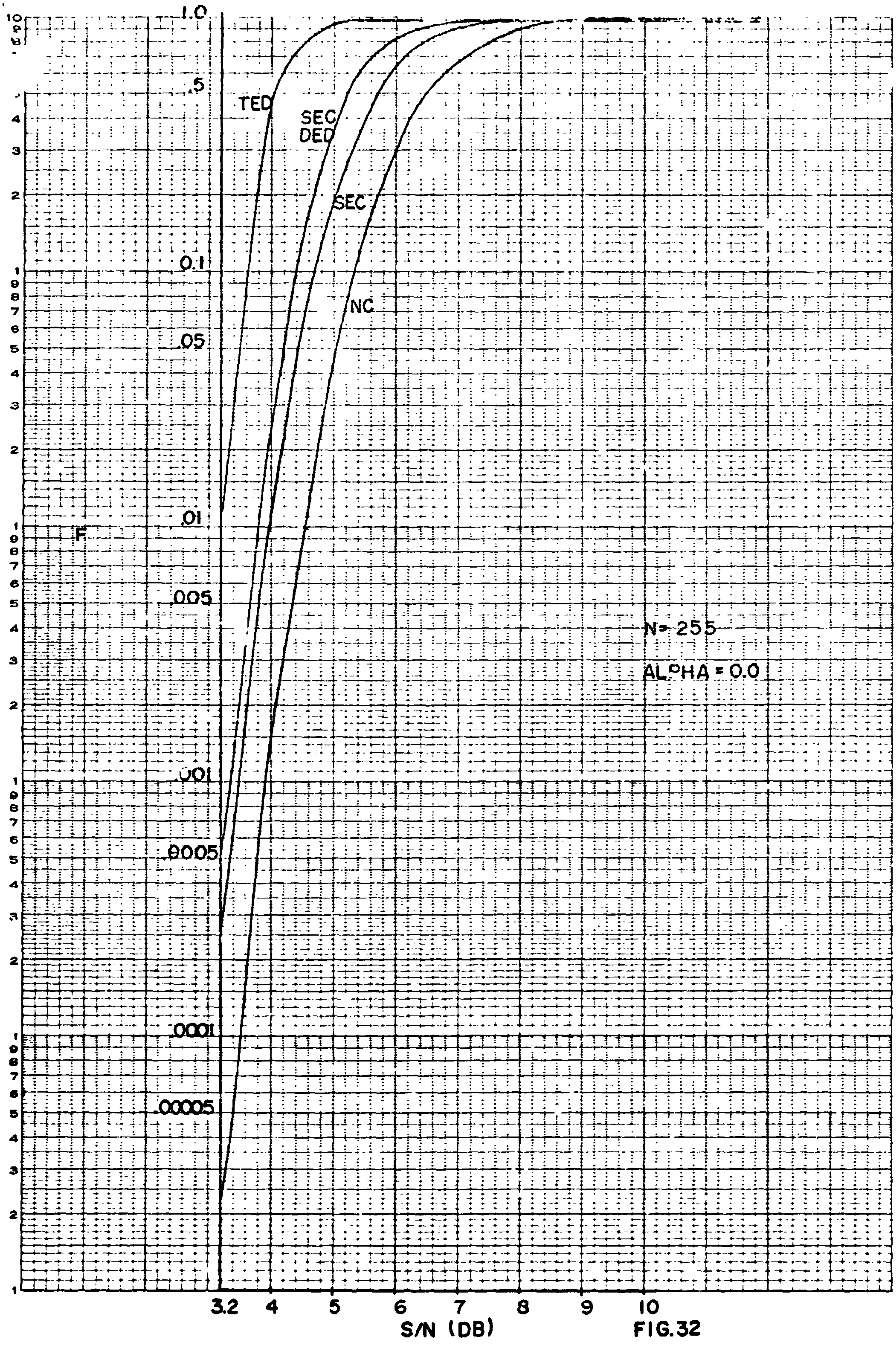
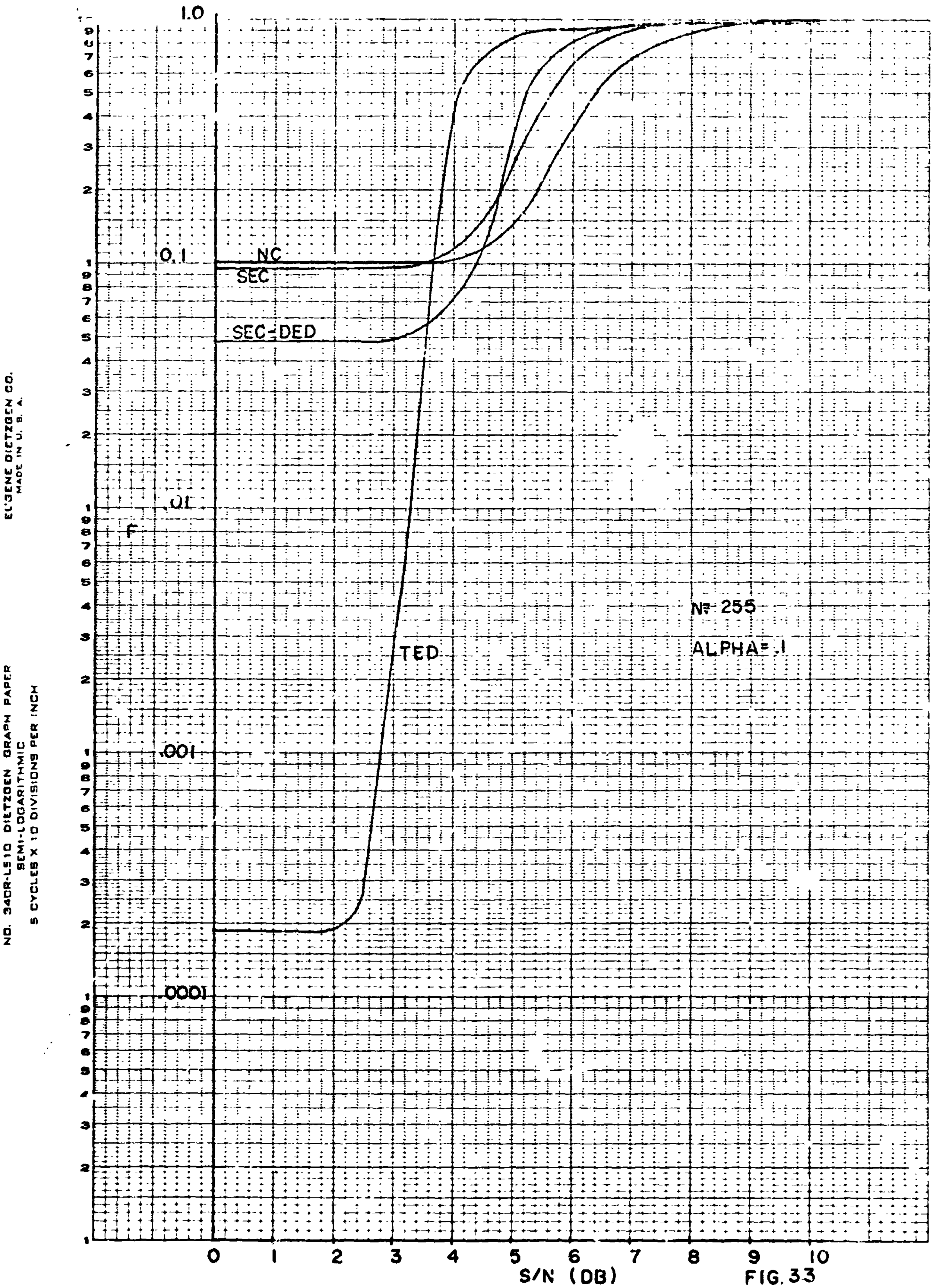

NC
SEC
SEC-DED
TED
N=127
ALPHA= 1.00
F
1.0
.5
.1
.05
.01
.005
S/N (DB)
NO. 340R-L310 DIETZGEN GRAPH PAPER
SEMI-LOGARITHMIC
3 CYCLES X 10 DIVISIONS PER INCH
EUGENE DIETZGEN CO.
MADE IN U. S. A.

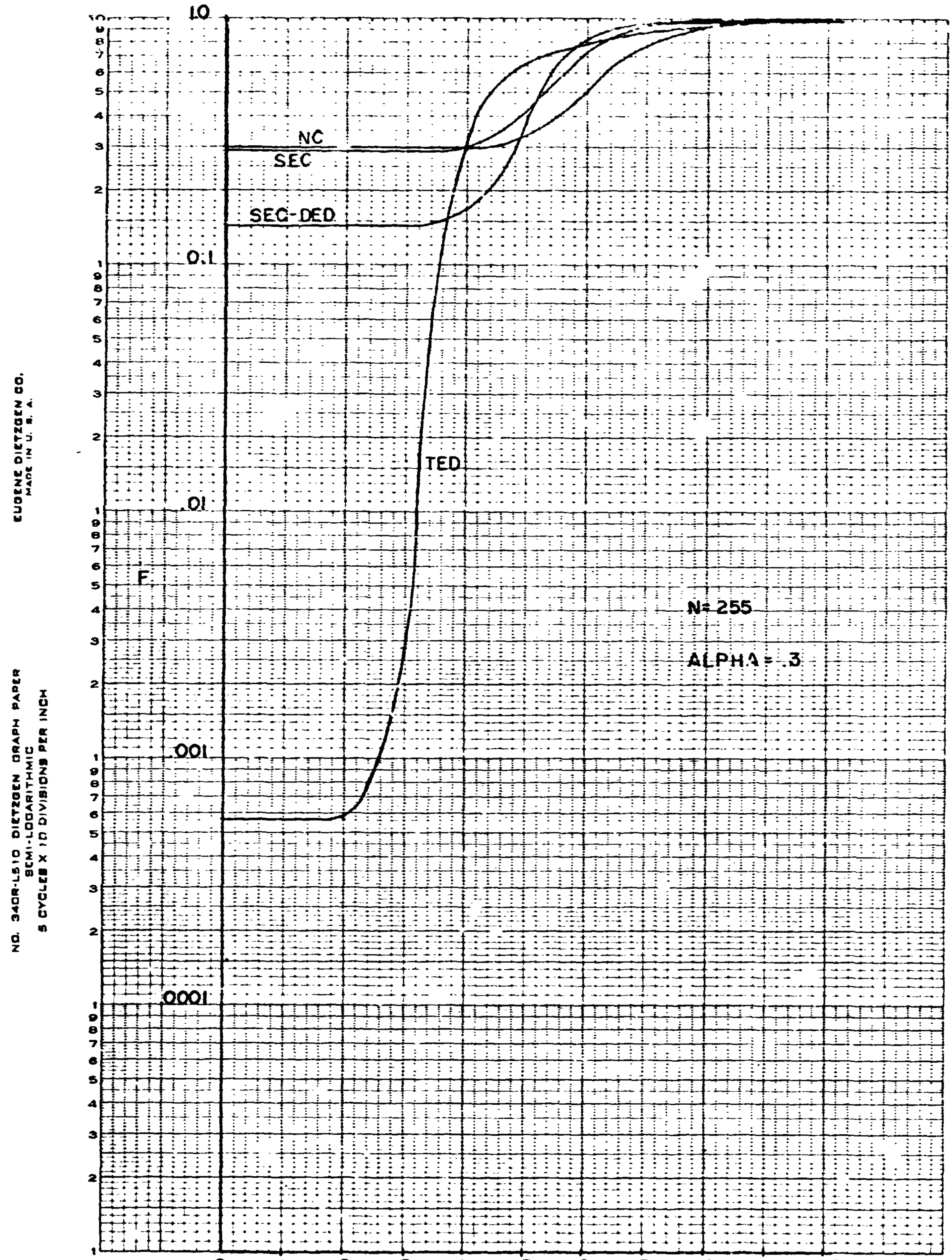
FIG. 31


1.0
.5
TED
SEC
DED
SEC
0.1
NC
.05
.01
F
.005
N= 255
ALPHA = 0.0
.001
.0005
.0001
.00005
3.2
4
5
6
7
8
9
10
S/N (DB)
FIG.32



1.0
0.1
.01
.001
.0001
F
NC
SEC
SEC-DED
TED
№ 255
ALPHA=.1
0
1
2
3
4
5
6
7
8
9
10
S/N (DB)
NO. 340R-L510 DIETZGEN GRAPH PAPER
SEMI-LOGARITHMIC
5 CYCLES X 10 DIVISIONS PER INCH
EUGENE DIETZGEN CO.
MADE IN U. S. A.

FIG. 33



1.0
NC
SEC
SEC-DED
0.1
TED
.01
F
N=255
ALPHA=.3
.001
.0001
0
1
2
3
4
5
6
7
8
9
10
S/N (DB)
EUGENE DIETZGEN CO.
MADE IN U. S. A.
NO. 340R-L510 DIETZGEN GRAPH PAPER
SEMI-LOGARITHMIC
5 CYCLES X 10 DIVISIONS PER INCH

FIG. 34

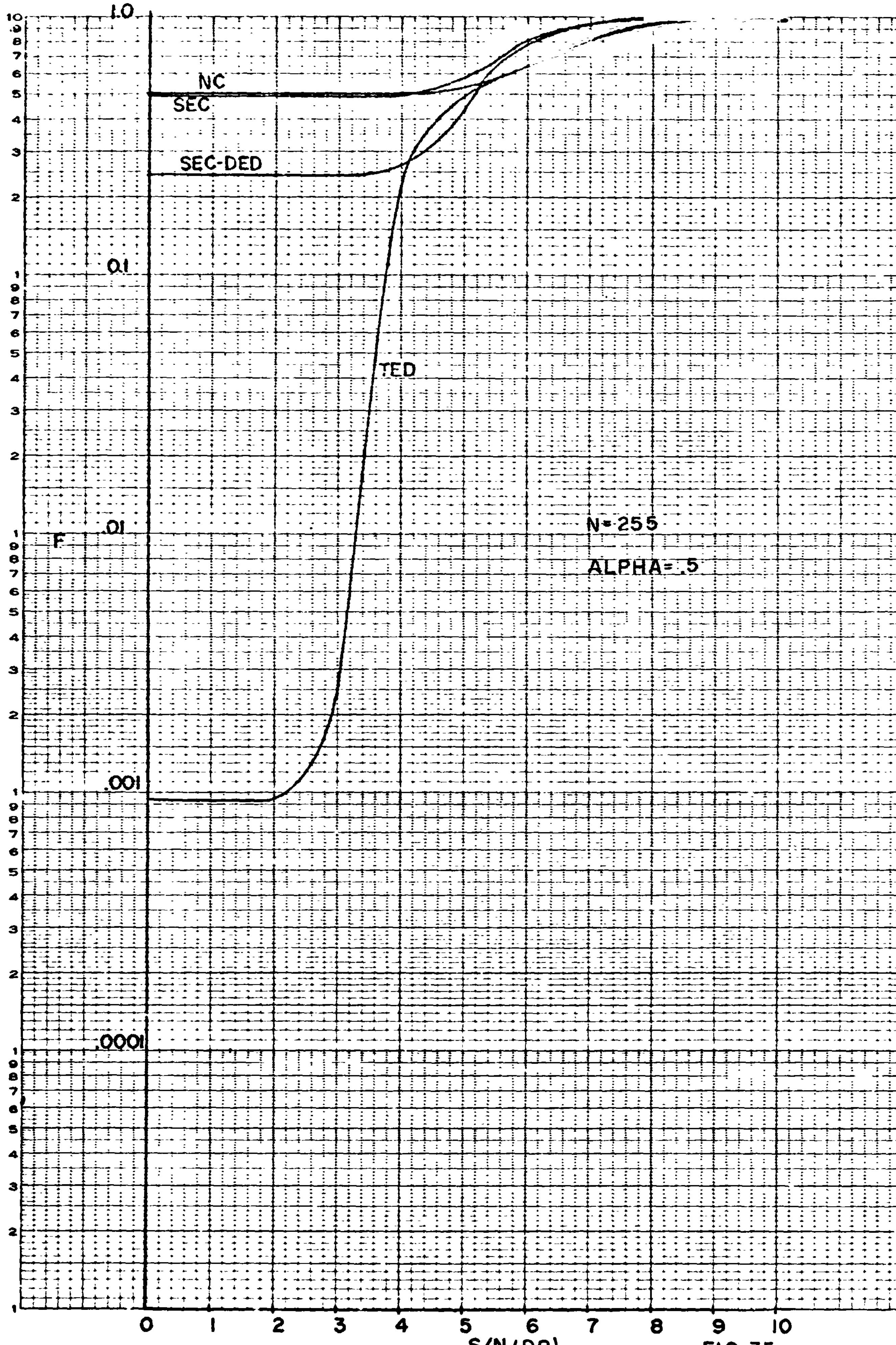

1.0
0.1
.01
.001
.0001
F
NC
SEC
SEC-DED
TED
N=255
ALPHA=.5
0 1 2 3 4 5 6 7 8 9 10
S/N (DB)

FIG.35

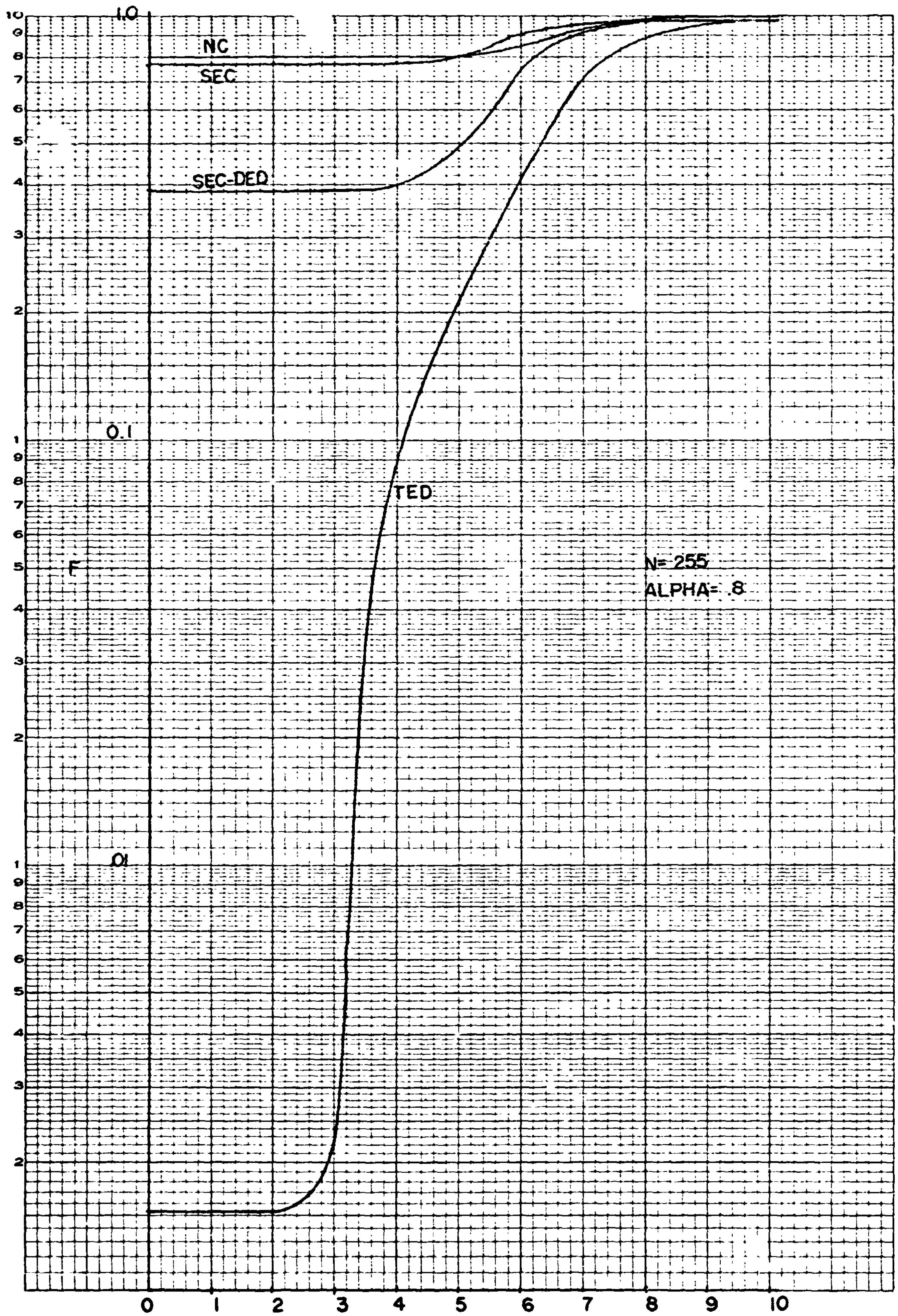
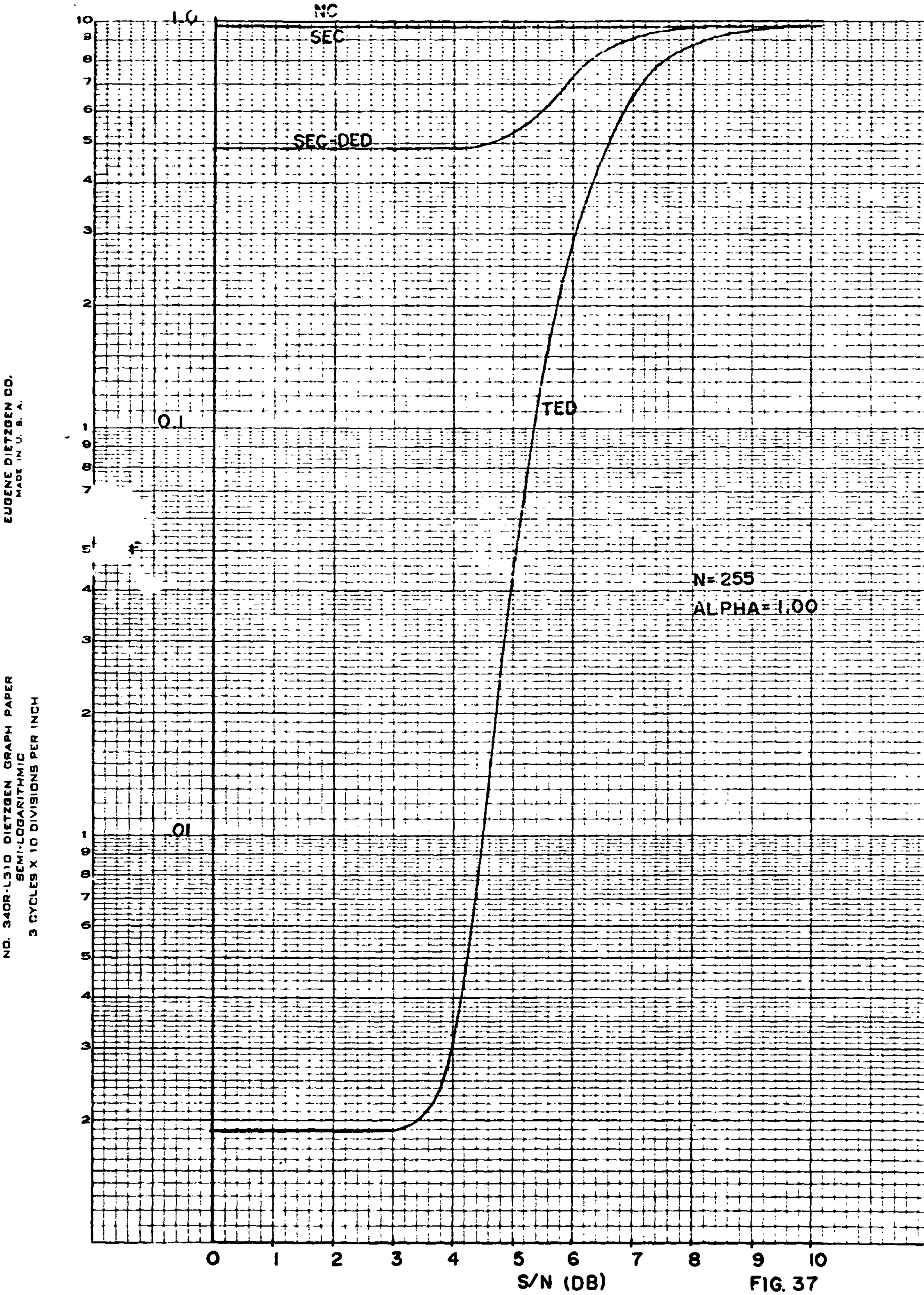
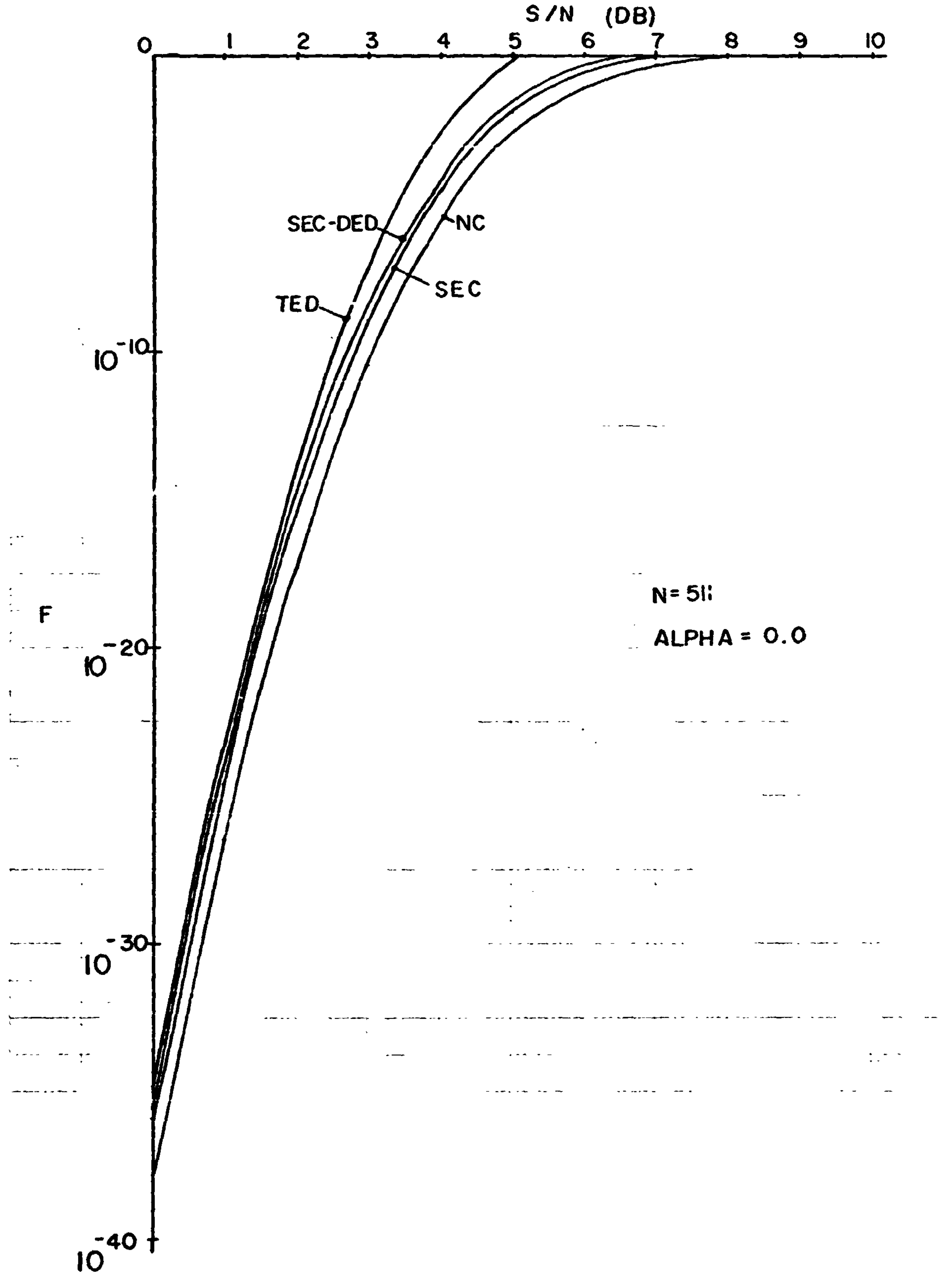


FIG. 36



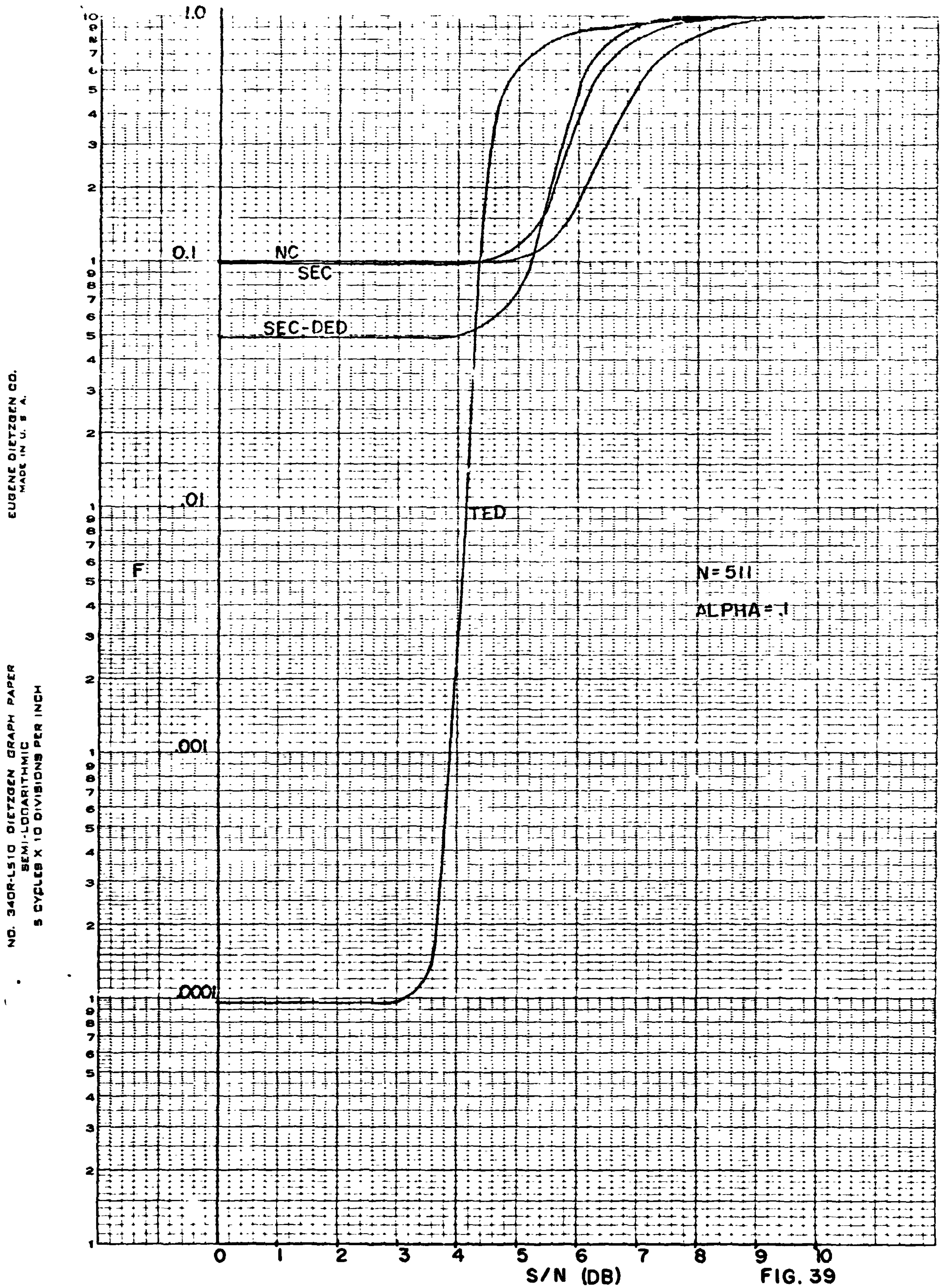
EUGENE DIETZGEN CO.
MADE IN U. S. A.
NO. 340R-L310 DIETZGEN GRAPH PAPER
SEMI-LOGARITHMIC
3 CYCLES X 10 DIVISIONS PER INCH
1.0
0.1
.01
NC
SEC
SEC-DED
TED
N=255
ALPHA=1.00
0
1
2
3
4
5
6
7
8
9
10
S/N (DB)

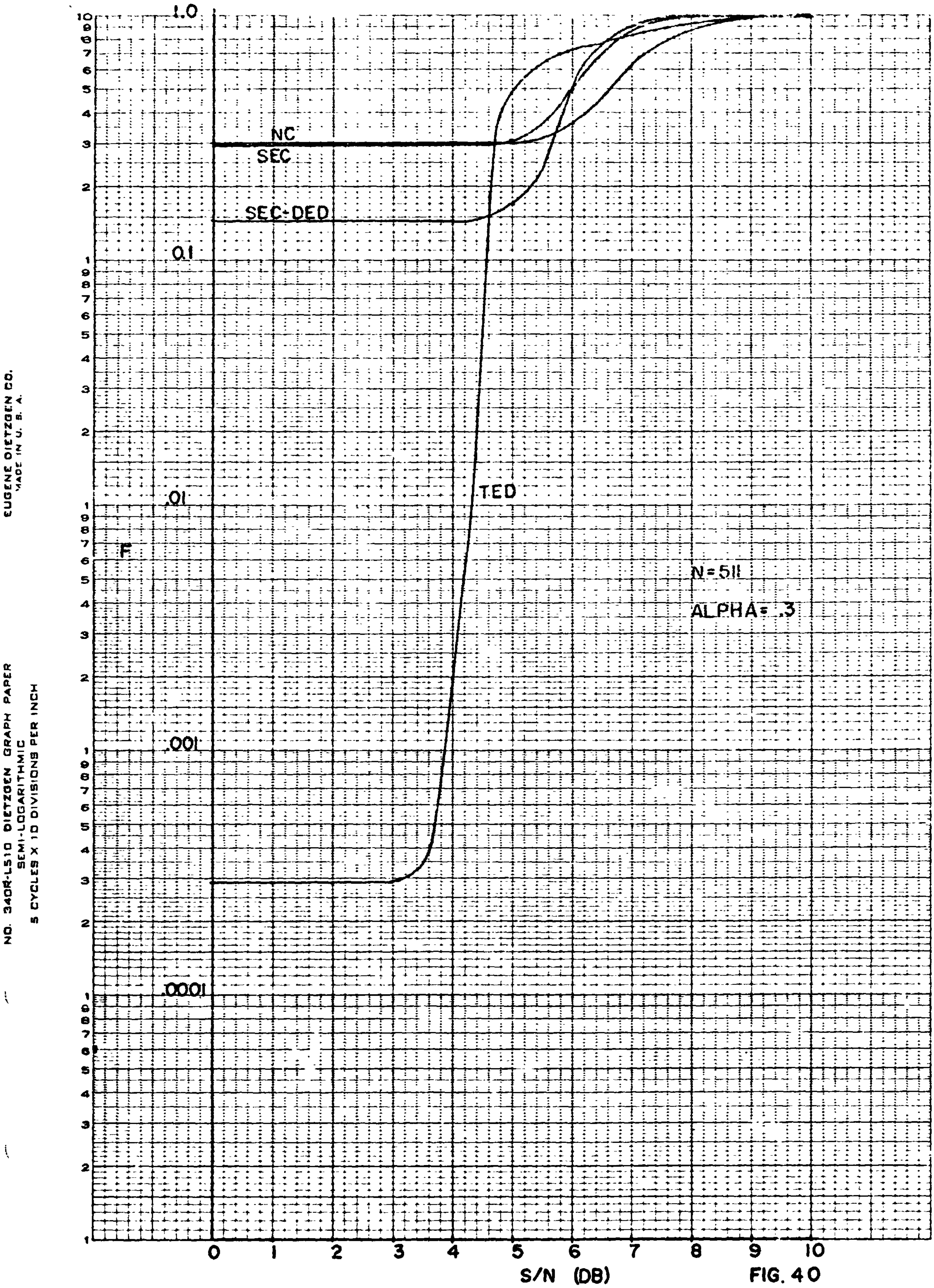
FIG. 37



S/N (DB)
0
1
2
3
4
5
6
7
8
9
10
SEC-DED
NC
SEC
TED
10^{-10}
F
10^{-20}
10^{-30}
10^{-40}
N=511
ALPHA = 0.0

FIG. 38


1.0
0.1
NC
SEC
SEC-DED
.01
TED
F
N=511
ALPHA=.1
.001
.0001
0 1 2 3 4 5 6 7 8 9 10
S/N (DB)
EUGENE DIETZGEN CO.
MADE IN U. S. A.
NO. 340R-L510 DIETZGEN GRAPH PAPER
SEMI-LOGARITHMIC
5 CYCLES X 10 DIVISIONS PER INCH
FIG. 39



1.0
0.1
.01
.001
.0001
F
NC
SEC
SEC+DED
TED
N=511
ALPHA=.3
0 1 2 3 4 5 6 7 8 9 10
S/N (DB)
NO. 340R-L510 DIETZGEN GRAPH PAPER
SEMI-LOGARITHMIC
5 CYCLES X 10 DIVISIONS PER INCH
EUGENE DIETZGEN CO.
MADE IN U. S. A.

FIG. 40

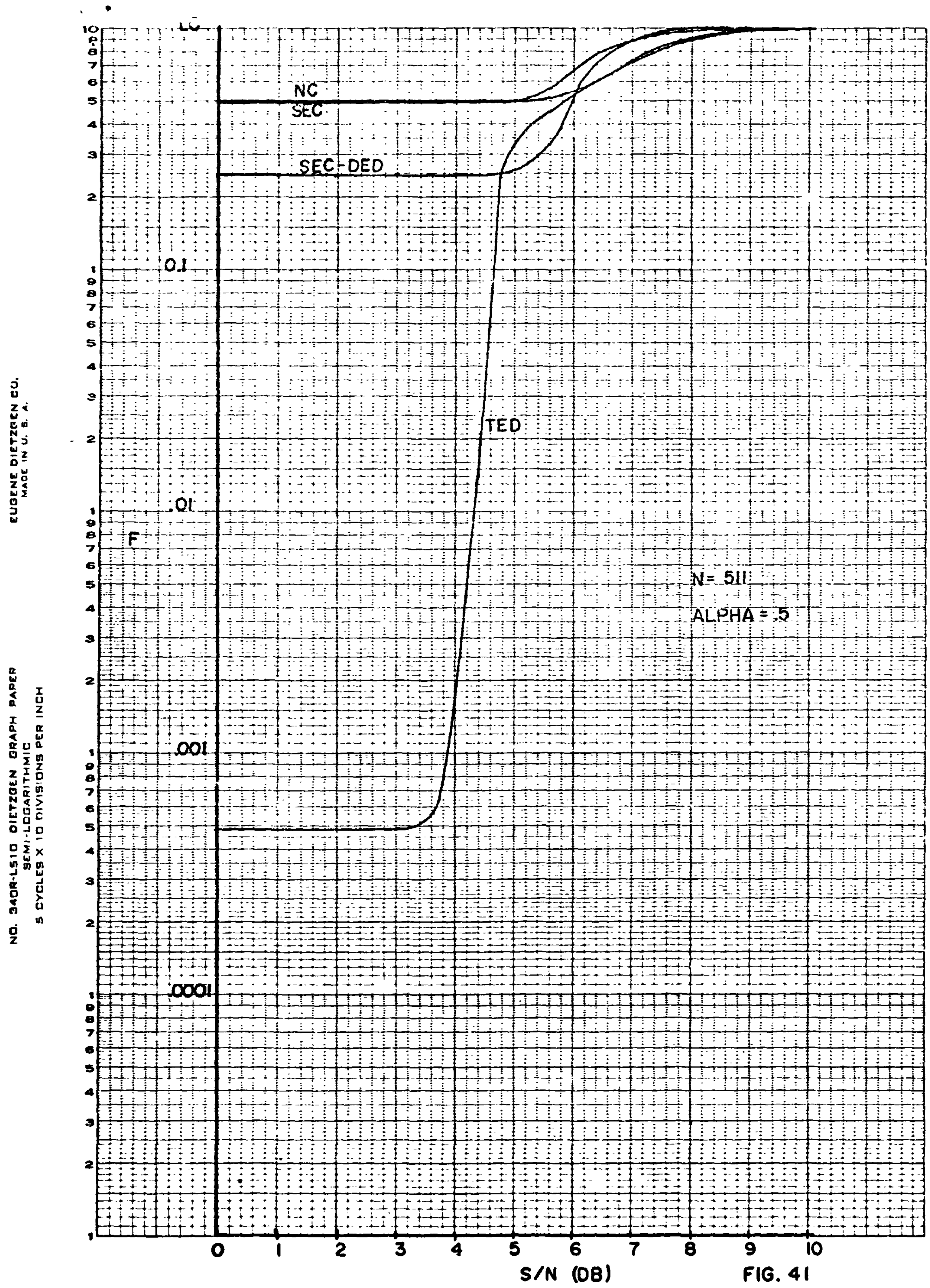
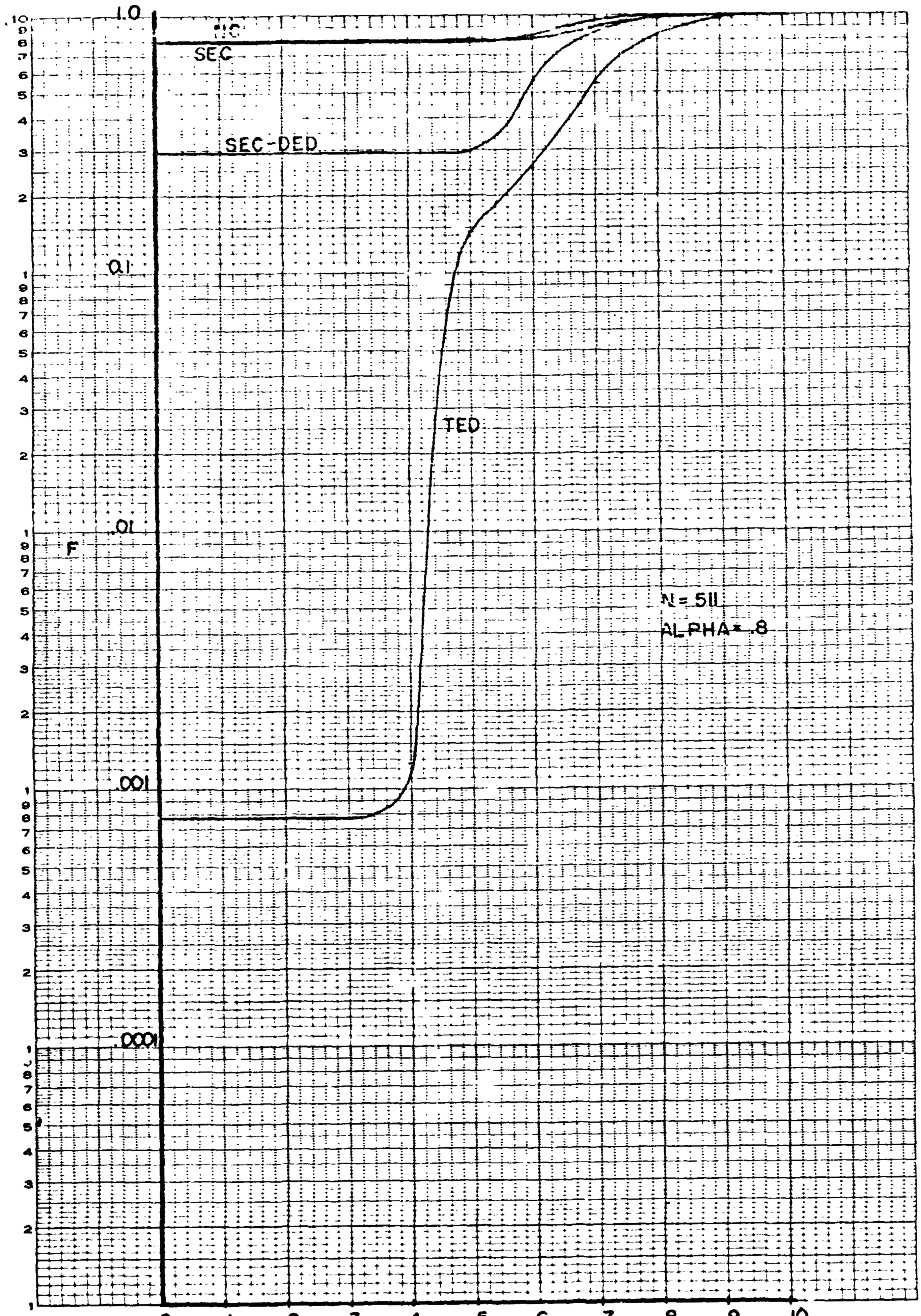

NC
SEC
SEC-DED
TED
F
0.1
.01
.001
.0001
N= 511
ALPHA = .5
0 1 2 3 4 5 6 7 8 9 10
S/N (DB)
EUGENE DIETZGEN CO.
MADE IN U. S. A.
NO. 340R-L510 DIETZGEN GRAPH PAPER
SEMI-LOGARITHMIC
5 CYCLES X 10 DIVISIONS PER INCH

FIG. 41



1.0
.01
.001
.0001
0.1
F
NC
SEC
SEC-DED
TED
N = 511
ALPHA = .8
0 1 2 3 4 5 6 7 8 9 10
S/N (DB)

FIG. 42

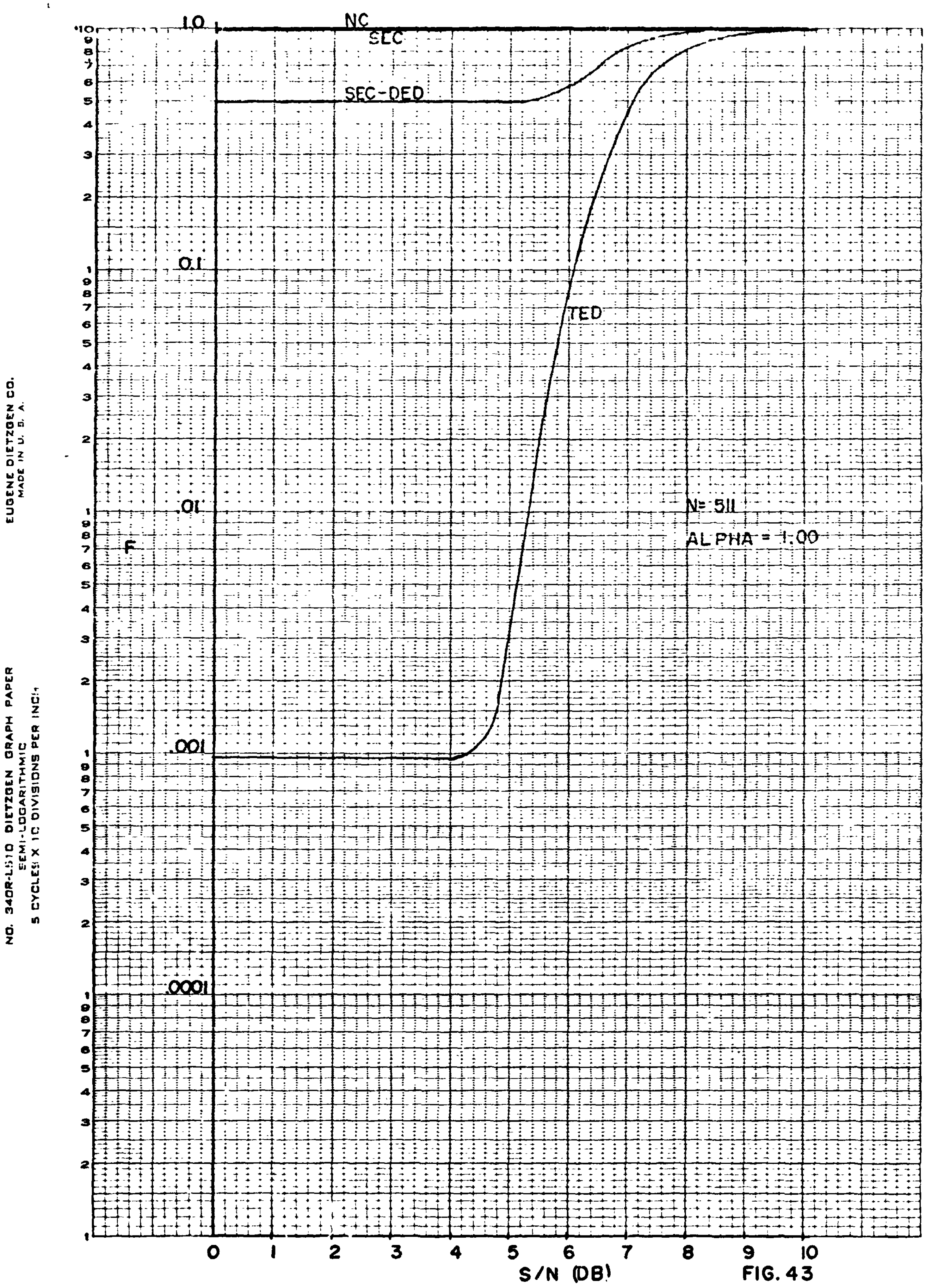
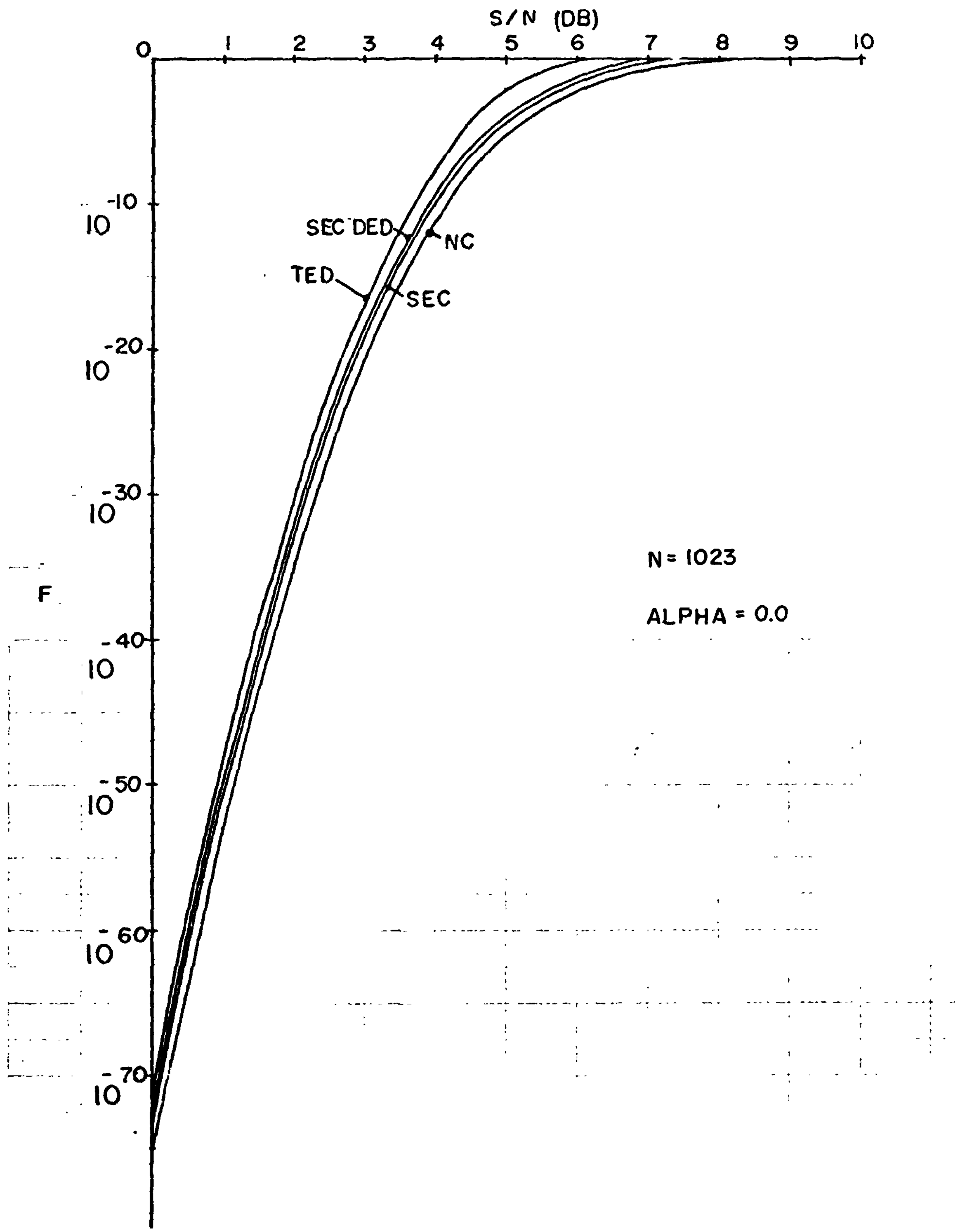

1.0
NC
SEC
SEC-DED
0.1
TED
.01
N=511
ALPHA = 1.00
F
.001
.0001
0
1
2
3
4
5
6
7
8
9
10
S/N (DB)
EUGENE DIETZGEN CO.
MADE IN U. S. A.
NO. 340R-L510 DIETZGEN GRAPH PAPER
SEMI-LOGARITHMIC
5 CYCLES X 10 DIVISIONS PER INCH

FIG. 43



S/N (DB)
0
1
2
3
4
5
6
7
8
9
10
10^{-10}
10^{-20}
10^{-30}
10^{-40}
10^{-50}
10^{-60}
10^{-70}
F
SEC DED
NC
TED
SEC
N = 1023
ALPHA = 0.0

FIG. 44

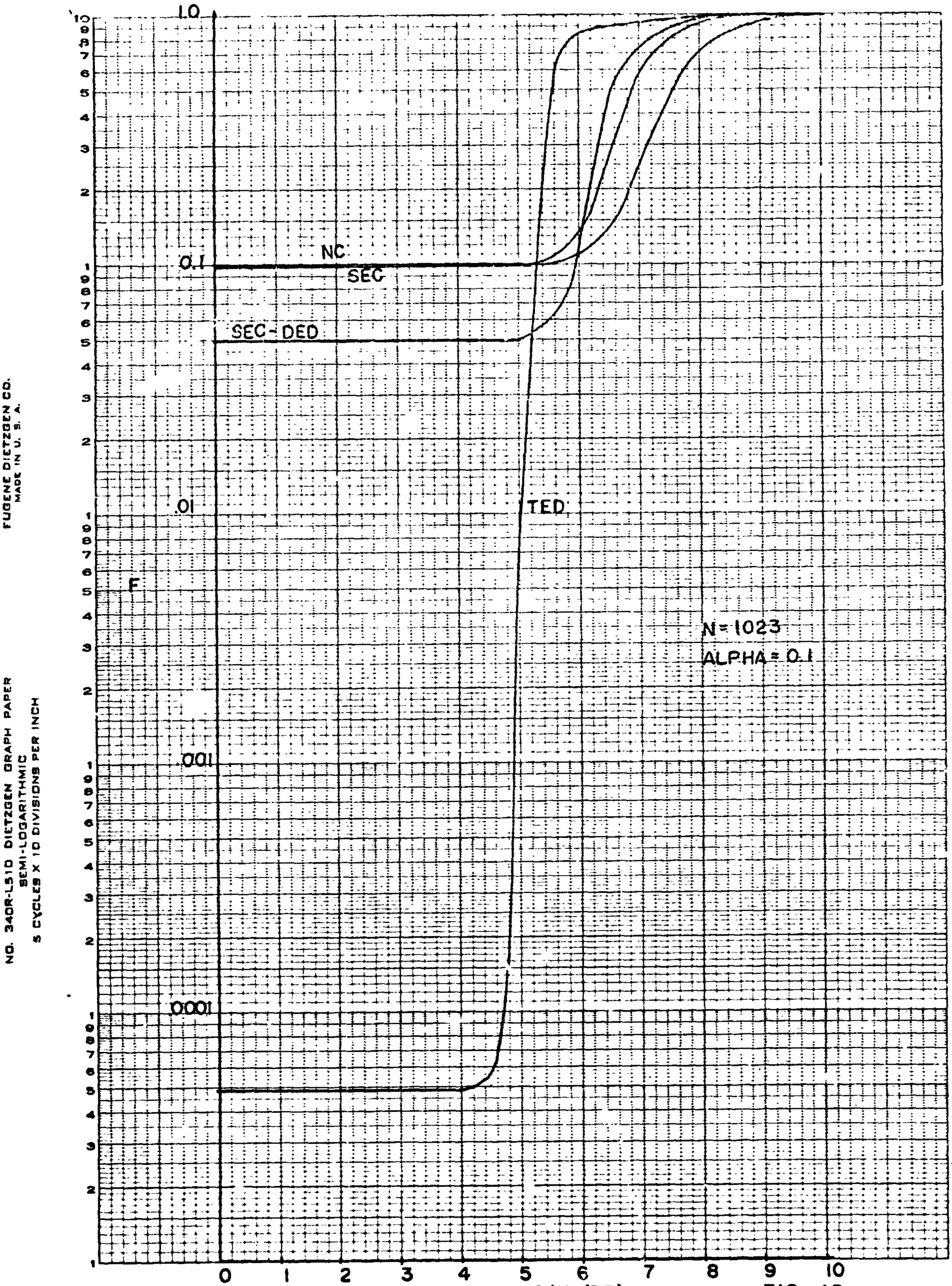
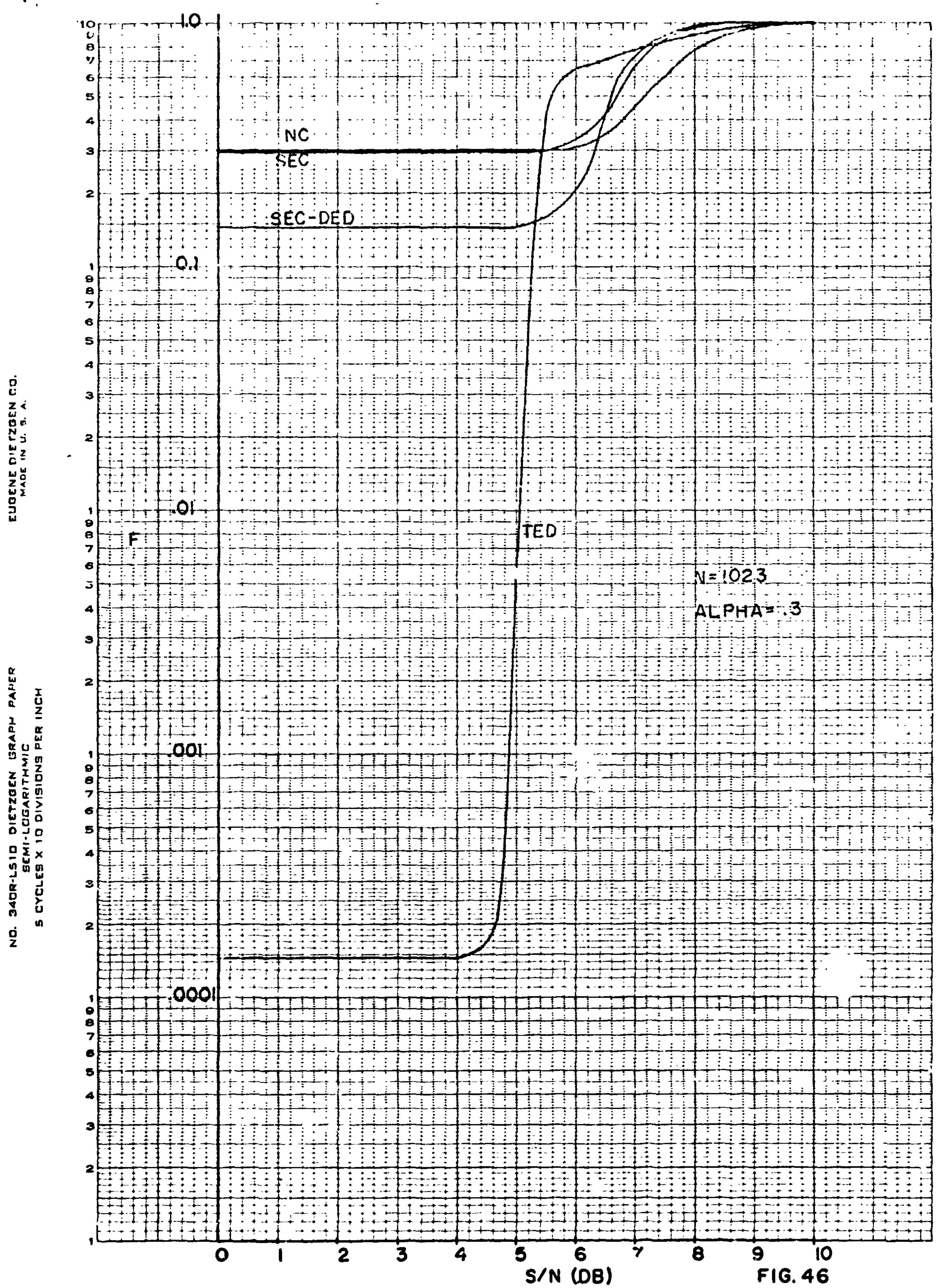

1.0
0.1
.01
.001
.0001
NC
SEC
SEC-DED
TED
F
N=1023
ALPHA= 0.1
0
1
2
3
4
5
6
7
8
9
10
S/N (DB)
NO. 340R-L510 DIETZGEN GRAPH PAPER
SEMI-LOGARITHMIC
5 CYCLES X 10 DIVISIONS PER INCH
EUGENE DIETZGEN CO.
MADE IN U. S. A.

FIG. 45



NO. 340R-L510 DIETZGEN GRAPH PAPER
SEMI-LOGARITHMIC
5 CYCLES X 10 DIVISIONS PER INCH
EUGENE DIETZGEN CO.
MADE IN U. S. A.
1.0
0.1
.01
.001
.0001
F
NC
SEC
SEC-DED
TED
N=1023
ALPHA=.3
0
1
2
3
4
5
6
7
8
9
10
S/N (DB)

FIG. 46

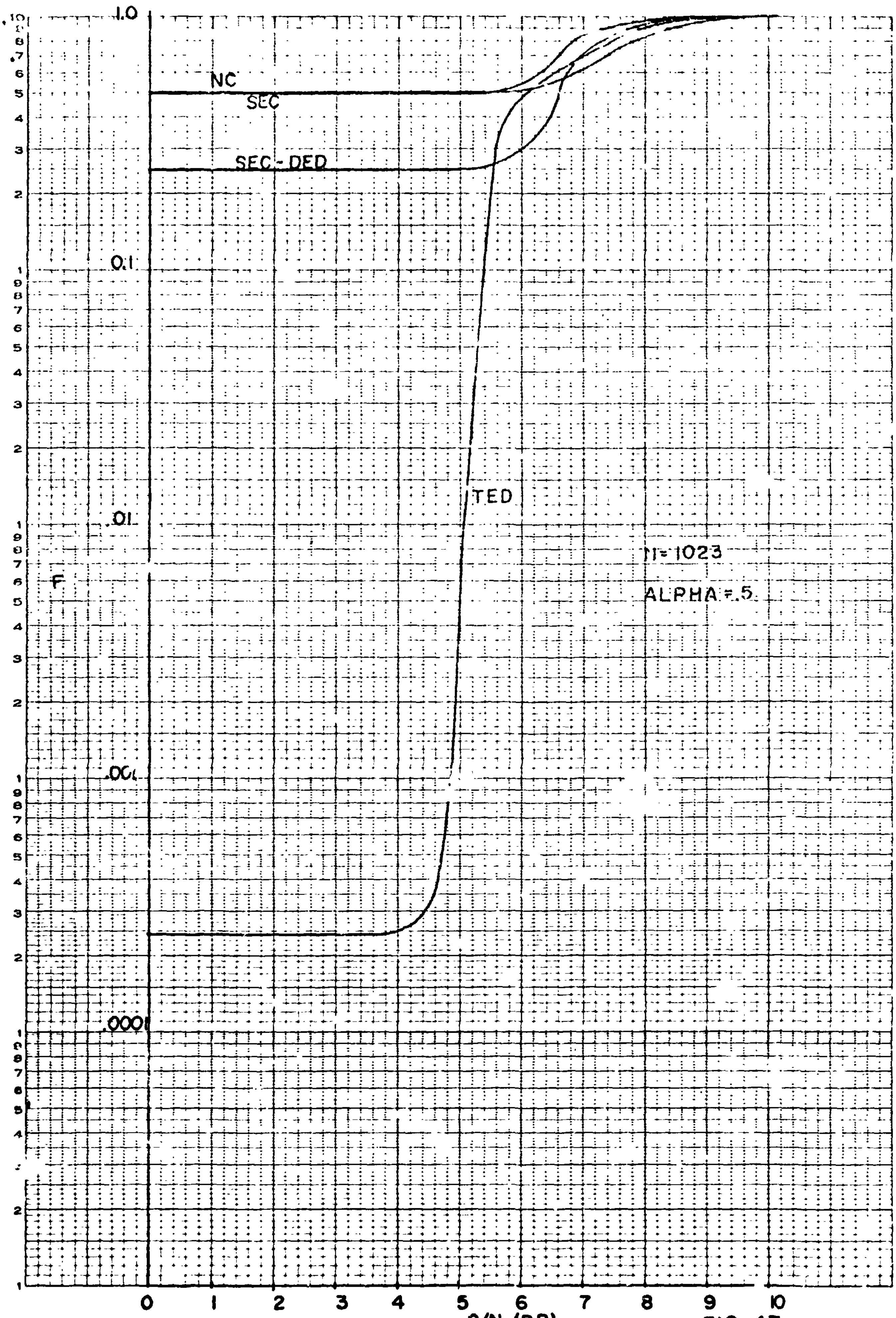

1.0
0.1
.01
.001
.0001
F
NC
SEC
SEC-DED
TED
n=1023
ALPHA=.5
0 1 2 3 4 5 6 7 8 9 10
S/N (DB)

FIG. 47

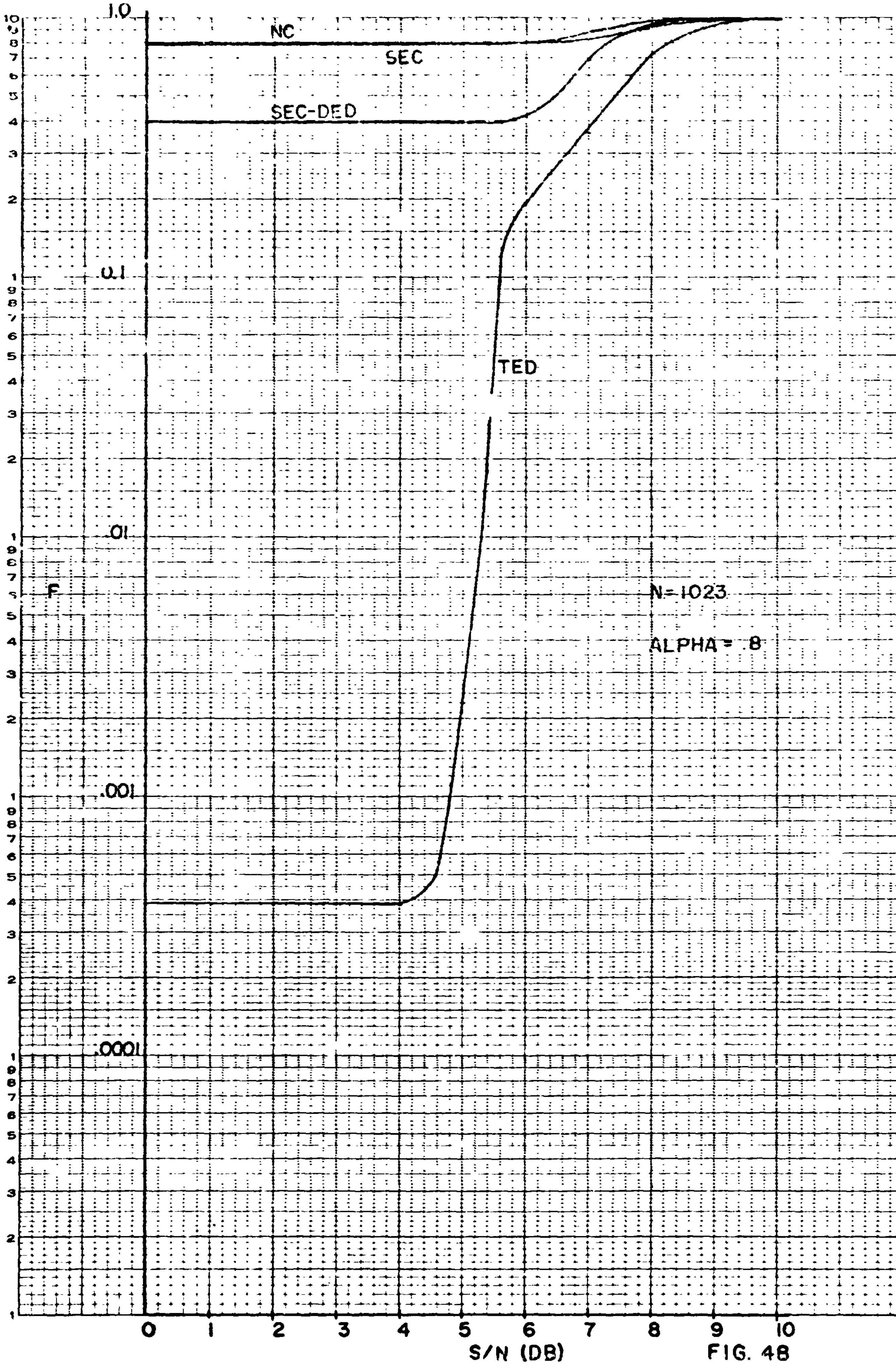
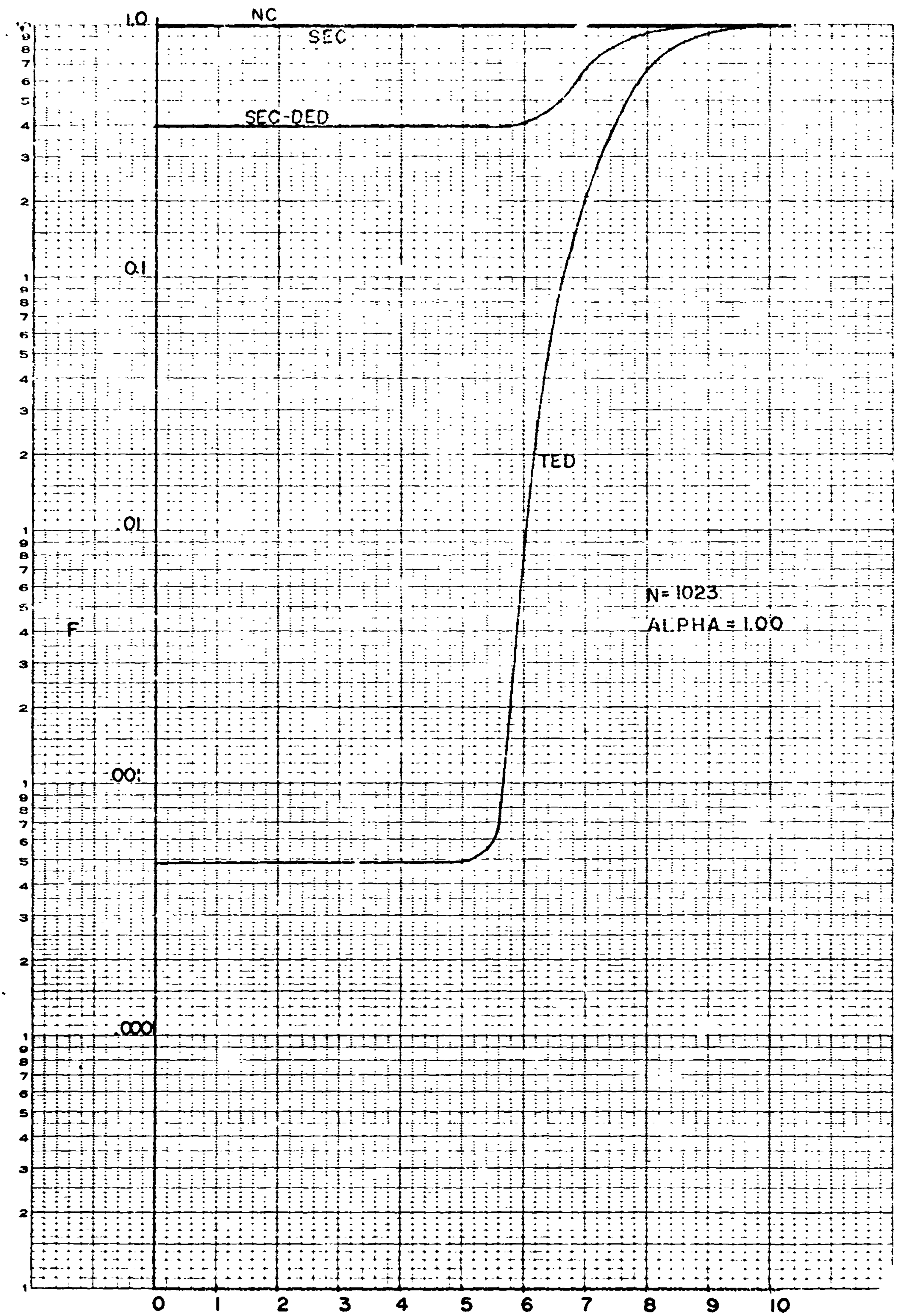

1.0
0.1
.01
.001
.0001
F
NC
SEC
SEC-DED
TED
N=1023
ALPHA = .8
0 1 2 3 4 5 6 7 8 9 10
S/N (DB)

FIG. 48



1.0
NC
SEC
SEC-DED
0.1
.01
TED
N= 1023
ALPHA = 1.00
F
.001
.0001
0
1
2
3
4
5
6
7
8
9
10
S/N (DB)

FIG. 49